

CLASSIFICATION OF SUBBUNDLES ON THE FARGUES-FONTAINE CURVE

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ABSTRACT. We completely classify all subbundles of a given vector bundle on the Fargues-Fontaine curve, where by a subbundle we mean a locally free subsheaf. Our classification is given in terms of a simple and explicit condition on Harder-Narasimhan polygons. Our proof is inspired by the proof of the main theorem in [Hon19], but also involves a number of nontrivial adjustments.

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1. INTRODUCTION

1.1. Motivation and background.

The *Fargues-Fontaine curve* is a regular Noetherian scheme of Krull dimension 1 which is constructed by Fargues-Fontaine [FF18] as the fundamental curve of p -adic Hodge theory. Powered by the theory of perfectoid spaces and diamonds as developed by Scholze in [Sch12] and [Sch18], the theory of vector bundles on the Fargues-Fontaine curve has driven a number of spectacular discoveries in arithmetic geometry and p -adic Hodge theory. Notable examples include the geometrization of the local Langlands correspondence by Fargues [Far16] and the construction of general local Shimura varieties by Scholze [SW].

One of the most fundamental results about vector bundles on the Fargues-Fontaine curve is that they form a slope category which admits a complete classification by Harder-Narasimhan (HN) polygons, as we briefly recall below.

Theorem 1.1.1 (Fargues-Fontaine [FF18], Kedlaya [Ked08]). *Fix a prime number p . Let E be a finite extension of \mathbb{Q}_p , and let F be an algebraically closed perfectoid field of characteristic p . Denote by $X = X_{E,F}$ the Fargues-Fontaine curve associated to the pair (E, F) .*

- (1) *The category of vector bundles on X admits a well-defined notion of slope.*
- (2) *For every rational number λ , there is a unique stable bundle of slope λ on X , denoted by $\mathcal{O}(\lambda)$.*
- (3) *Every semistable bundle on X of slope λ is of the form $\mathcal{O}(\lambda)^{\oplus m}$.*
- (4) *Every vector bundle \mathcal{V} on X admits a (necessarily unique) Harder-Narasimhan decomposition*

$$\mathcal{V} \simeq \bigoplus_i \mathcal{O}(\lambda_i)^{\oplus m_i}.$$

In other words, the isomorphism class of \mathcal{V} is determined by the Harder-Narasimhan polygon $\text{HN}(\mathcal{V})$ of \mathcal{V} .

Theorem 1.1.1 naturally leads to a question of classifying all quotient bundles and subbundles of a given vector bundle on the Fargues-Fontaine curve. For quotient bundles, the author in [Hon19] has obtained a complete classification in terms of HN polygons. Our main purpose in this paper is to obtain a complete classification for subbundles.

We remark that the classification problem for subbundles is closely related to the study of *modifications of vector bundles*, which play a pivotal role in studying the geometry of the B_{dR} -affine Grassmannians, the flag varieties, and the Hecke stacks. By definition, a modification of vector bundles is an exact sequence of the form

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{T} \longrightarrow 0$$

for some vector bundles \mathcal{E}, \mathcal{F} and torsion sheaf \mathcal{T} . When \mathcal{F} is fixed, the vector bundles that can take place of \mathcal{E} are precisely subbundles of \mathcal{F} with maximal rank; therefore, once we have a complete classification for subbundles of \mathcal{F} , we can describe all possible isomorphism classes of \mathcal{E} .

1.2. Overview of the result.

For a vector bundle \mathcal{V} on X and a rational number μ , we define a vector bundle $\mathcal{V}^{\geq \mu}$ by declaring that its HN polygon $\text{HN}(\mathcal{V}^{\geq \mu})$ consists of all line segments in $\text{HN}(\mathcal{V})$ with slope greater than or equal to μ . In other words, for a vector bundle \mathcal{V} on X with HN decomposition

$$\mathcal{V} \simeq \bigoplus_i \mathcal{O}(\lambda_i)^{\oplus m_i},$$

we set

$$\mathcal{V}^{\geq \mu} := \bigoplus_{\lambda_i \geq \mu} \mathcal{O}(\lambda_i)^{\oplus m_i} \quad \text{for every } \mu \in \mathbb{Q}.$$

We can state our main result as follows:

Theorem 1.2.1. *Let \mathcal{F} be a vector bundle on X . Then a vector bundle \mathcal{E} on X is a subbundle of \mathcal{F} (i.e., a locally free subsheaf of \mathcal{F}) if and only if the following equivalent conditions are satisfied:*

- (i) *$\text{rank}(\mathcal{E}^{\geq \mu}) \leq \text{rank}(\mathcal{F}^{\geq \mu})$ for every $\mu \in \mathbb{Q}$.*
- (ii) *For each $i = 1, \dots, \text{rank}(\mathcal{E})$, the slope of $\text{HN}(\mathcal{E})$ on the interval $[i-1, i]$ is less than or equal to the slope of $\text{HN}(\mathcal{F})$ on this interval.*

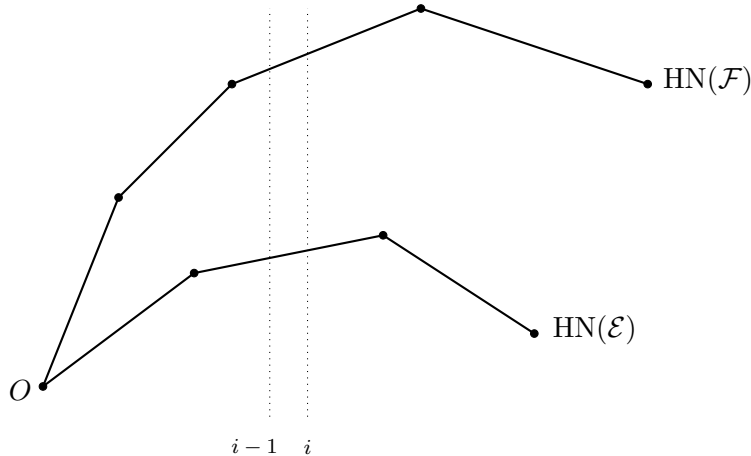


FIGURE 1. Illustration of the condition (ii) in Theorem 3.1.1.

Let us briefly sketch our proof of Theorem 1.2.1, which is largely inspired by the main argument in [Hon19] for classification of quotient bundles. The necessity part of Theorem 1.2.1 is a direct consequence of the slope formalism, while equivalence of the conditions (i) and (ii) follows immediately from convexity of HN polygons. Thus the main part of the proof will concern the sufficiency part of Theorem 1.2.1. To this end, we will consider auxiliary moduli spaces $\mathcal{H}om(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}$ which (roughly) parametrize bundle maps $\mathcal{E} \rightarrow \mathcal{F}$ with image isomorphic to a specified vector bundle \mathcal{Q} . These spaces are *diamonds* in the sense of Scholze [Sch18], as shown in [BFH⁺17]. For the assertion that \mathcal{E} is a subbundle of \mathcal{F} , it suffices to prove nonemptiness of $\mathcal{H}om(\mathcal{E}, \mathcal{F})_{\mathcal{E}}$. Using the dimension theory for diamonds, we will reduce the desired nonemptiness of $\mathcal{H}om(\mathcal{E}, \mathcal{F})_{\mathcal{E}}$ to a quantitative statement as stated in Proposition 3.2.1. Then we will prove this quantitative statement by a certain degenerating process on the dual bundle of \mathcal{E} .

In many parts, our argument will adapt various notions and constructions from [Hon19]. Most notably, the notion of *slopewise dominance* as defined in [Hon19] will play a crucial role in both the formulation and the proof of the key quantitative statement, namely Proposition 3.2.1. In addition, our degenerating process on the dual bundle of \mathcal{E} will be almost identical to the degenerating process on \mathcal{F} in the main argument of [Hon19].

However, the details of our argument will require several nontrivial adjustments from the argument in [Hon19]. These adjustments essentially come from the fact that Theorem 1.2.1 cannot be deduced by simply dualizing the classification theorem for quotient bundles as obtained in [Hon19]. In our proof, we will try to indicate what adjustments we should make and why such adjustments are necessary.

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2. VECTOR BUNDLES ON THE FARGUES-FONTAINE CURVE

2.1. The Fargues-Fontaine curve.

Throughout this paper, we fix the following data:

- p is a prime number;
- E is a finite extension of \mathbb{Q}_p with residue field \mathbb{F}_q ;
- F is an algebraically closed perfectoid field of characteristic p .

In addition, we denote by E° and F° the rings of integers of E and F , respectively. We also choose a uniformizer π of E and a pseudouniformizer ϖ of F .

Let $W_{E^\circ}(F^\circ) := W(F^\circ) \otimes_{W(\mathbb{F}_q)} E^\circ$ be the ring of ramified Witt vectors of F° with coefficients in E° , and let $[\varpi]$ be the Teichmuller lift of ϖ . One can show that

$$\mathcal{Y} := \mathrm{Spa}(W_{E^\circ}(F^\circ)) \setminus \{|p[\varpi]| = 0\}$$

is an adic space over $\mathrm{Spa}(E)$. Moreover, the natural q -Frobenius map on $W_{E^\circ}(F^\circ)$ induces a properly discontinuous automorphism ϕ of \mathcal{Y} .

Definition 2.1.1. Given the pair (E, F) , we define the associated *adic Fargues-Fontaine curve* by

$$\mathcal{X} := \mathcal{Y}/\phi^{\mathbb{Z}},$$

and the associated *schematic Fargues-Fontaine curve* by

$$X := \mathrm{Proj} \left(\bigoplus_{n \geq 0} H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})^{\phi = \pi^n} \right).$$

In this paper, we will speak interchangeably about vector bundles on \mathcal{X} and X . There will be no harm from doing this because of the following GAGA type result:

Proposition 2.1.2 (“GAGA for the Fargues-Fontaine curve”, [KL15, Theorem 6.3.12]). *There is a natural map of locally ringed spaces*

$$\mathcal{X} \rightarrow X$$

which induces by pullback an equivalence of categories of vector bundles.

The Fargues-Fontaine curve is a “curve” in the following sense:

Proposition 2.1.3 ([FF18]). *The scheme X is a regular, Noetherian scheme over E of Krull dimension 1.*

Remark. However, the scheme X is not of finite type over E . In fact, the residue field at a closed point is a complete algebraically closed extension of E .

We can extend the construction of the Fargues-Fontaine curve to relative settings. Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathrm{Spa}(F)$, and let ϖ_R be a pseudouniformizer of R . We take the ring of ramified Witt vectors $W_{E^\circ}(R^+) := W(R^+) \otimes_{W(\mathbb{F}_q)} E^\circ$ and write $[\varpi_R]$ for the Teichmuller lift of ϖ_R . One can show that

$$\mathcal{Y}_S := \mathrm{Spa}(W_{E^\circ}(R^+), W_{E^\circ}(R^+)) \setminus \{|p[\varpi_R]| = 0\}$$

is an adic space over $\mathrm{Spa}(E)$, equipped with a properly discontinuous automorphism ϕ induced by the natural q -Frobenius on $W_{E^\circ}(R^+)$.

Definition 2.1.4. Given an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ over $\mathrm{Spa}(F)$, we define the *adic Fargues-Fontaine curve* associated to the pair (E, S) by

$$\mathcal{X}_S := \mathcal{Y}_S / \phi^{\mathbb{Z}},$$

and the *schematic Fargues-Fontaine curve* associated to the pair (E, S) by

$$X_S := \mathrm{Proj} \left(\bigoplus_{n \geq 0} H^0(\mathcal{Y}_S, \mathcal{O}_{\mathcal{Y}})^{\phi = \pi^n} \right).$$

More generally, for an arbitrary perfectoid space S over $\mathrm{Spa}(F)$, we choose an affinoid cover $S = \bigcup S_i = \bigcup \mathrm{Spa}(R_i, R_i^+)$ and define the adic Fargues-Fontaine curve \mathcal{X}_S and the schematic Fargues-Fontaine curve X_S respectively by gluing the \mathcal{X}_{S_i} and the X_{S_i} .

There is a GAGA type result which extends Proposition 2.1.2 to relative settings, thereby allowing us to speak interchangeably about vector bundles on \mathcal{X}_S and X_S for any perfectoid space S over $\mathrm{Spa}(F)$.

2.2. Slope theory for vector bundles.

In this subsection we briefly review the slope theory for vector bundles on the Fargues-Fontaine curve.

Definition 2.2.1. Given a vector bundle \mathcal{V} on X , we write $\mathrm{rk}(\mathcal{V})$ for the rank of \mathcal{V} and \mathcal{V}^\vee for the dual bundle of \mathcal{V} .

The Fargues-Fontaine curve X is not a complete curve; in fact, as remarked after Proposition 2.1.3, it is not even of finite type. Nonetheless, the Fargues-Fontaine curve behaves as a complete curve in the following sense:

Proposition 2.2.2 ([FF18]). *For an arbitrary nonzero rational function f on X , its divisor $\mathrm{div}(f)$ has degree zero.*

We thus have a well-defined notion of degree and slope for vector bundles on X .

Definition 2.2.3. Let \mathcal{V} be a vector bundle on X .

- (1) If \mathcal{V} is a line bundle (i.e., $\mathrm{rk}(\mathcal{V}) = 1$), we define the *degree* of \mathcal{V} by

$$\mathrm{deg}(\mathcal{V}) := \mathrm{deg}(\mathrm{div}(s))$$

where s is an arbitrary nonzero meromorphic section of \mathcal{V} . In general, we define

$$\mathrm{deg}(\mathcal{V}) := \mathrm{deg}(\wedge^{\mathrm{rk}(\mathcal{V})} \mathcal{V}).$$

- (2) We define the *slope* of \mathcal{V} by

$$\mu(\mathcal{V}) := \frac{\mathrm{deg}(\mathcal{V})}{\mathrm{rk}(\mathcal{V})}.$$

We explicitly construct some vector bundles on X which will serve as building blocks for general vector bundles on X . Let $\lambda = r/s$ be a rational number written in lowest terms with $r > 0$. We choose a trivializing basis v_1, v_2, \dots, v_s of $\mathcal{O}_{\mathcal{Y}}^{\oplus s}$, and define an isomorphism $\phi^* \mathcal{O}_{\mathcal{Y}}^{\oplus s} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}}^{\oplus s}$ by

$$v_1 \mapsto v_2, \quad v_2 \mapsto v_3, \quad \dots, \quad v_{s-1} \mapsto v_s, \quad v_s \mapsto \pi^{-r} v_1,$$

where we abuse notation to view v_1, v_2, \dots, v_s as a trivializing basis for $\phi^* \mathcal{O}_{\mathcal{Y}}^{\oplus s}$ as well. We denote by $\widetilde{\mathcal{O}}(\lambda)$ the vector bundle $\mathcal{O}_{\mathcal{Y}}^{\oplus s}$ equipped with the isomorphism $\phi^* \mathcal{O}_{\mathcal{Y}}^{\oplus s} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}}^{\oplus s}$ as defined above.

Definition 2.2.4. Given a rational number λ , we write $\mathcal{O}(\lambda)$ for the vector bundle on \mathcal{X} obtained by descending the vector bundle $\widehat{\mathcal{O}(\lambda)}$, and also for the corresponding vector bundle on X under the GAGA functor described in Proposition 2.1.2.

Lemma 2.2.5. *Let $\lambda = r/s$ be a rational number written in lowest terms with $r > 0$.*

- (1) $\text{rk}(\mathcal{O}(\lambda)) = s$ and $\text{deg}(\mathcal{O}(\lambda)) = r$.
- (2) $\mathcal{O}(\lambda)^\vee \simeq \mathcal{O}(-\lambda)$.

Proof. All statements are straightforward to check using Definition 2.2.4. \square

Let us now recall the notions of semistability and stability.

Definition 2.2.6. A vector bundle \mathcal{V} on X is *semistable* if every nonzero proper subbundle \mathcal{W} of \mathcal{V} satisfies

$$\mu(\mathcal{W}) \leq \mu(\mathcal{V}). \quad (2.1)$$

A semistable vector bundle \mathcal{V} on X is *stable* if the equality in (2.1) never holds.

It turns out that the category of vector bundles on X admits an explicit characterization of stability and semistability, as well as a complete classification of isomorphism classes, in terms of the vector bundles that we constructed in Definition 2.2.4.

Theorem 2.2.7 ([FF18]). *Let \mathcal{V} be a vector bundle on X .*

- (1) \mathcal{V} is stable of slope λ if and only if $\mathcal{V} \simeq \mathcal{O}(\lambda)$.
- (2) \mathcal{V} is semistable of slope λ if and only if $\mathcal{V} \simeq \mathcal{O}(\lambda)^{\oplus n}$ for some n .
- (3) In general, \mathcal{V} admits a unique direct sum decomposition of the form

$$\mathcal{V} \simeq \bigoplus_{i=1}^l \mathcal{O}(\lambda_i)^{\oplus m_i} \quad (2.2)$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_l$.

Definition 2.2.8. Let \mathcal{V} be a vector bundle on X .

- (1) We refer to the decomposition (2.2) in Theorem 2.2.7 as the *Harder-Narasimhan (HN) decomposition* of \mathcal{V} .
- (2) We refer to the slopes λ_i of direct summands in the HN decomposition as the *Harder-Narasimhan (HN) slopes* of \mathcal{V} , or often simply as the *slopes* of \mathcal{V} .
- (3) We write $\mu_{\max}(\mathcal{V})$ (resp. $\mu_{\min}(\mathcal{V})$) for the maximum (resp. minimum) HN slope of \mathcal{V} . In other words, we set

$$\mu_{\max}(\mathcal{V}) := \lambda_1 \quad \text{and} \quad \mu_{\min}(\mathcal{V}) := \lambda_l.$$

- (4) For every $\mu \in \mathbb{Q}$ we set

$$\mathcal{V}^{\geq \mu} := \bigoplus_{\lambda_i \geq \mu} \mathcal{O}(\lambda_i)^{\oplus m_i} \quad \text{and} \quad \mathcal{V}^{\leq \mu} := \bigoplus_{\lambda_i \leq \mu} \mathcal{O}(\lambda_i)^{\oplus m_i},$$

and similarly define $\mathcal{V}^{> \mu}$ and $\mathcal{V}^{< \mu}$.

- (5) We define the *Harder-Narasimhan (HN) polygon* of \mathcal{V} as the upper convex hull of the points $(0, 0)$ and $(\text{rk}(\mathcal{V}^{\geq \lambda_i}), \text{deg}(\mathcal{V}^{\geq \lambda_i}))$

Remark. In the subsequent sections, we will often write $\text{deg}(\mathcal{V})^{\geq 0}$ in lieu of $\text{deg}(\mathcal{V}^{\geq 0})$. This will be particularly useful as we will often consider vector bundles of the form $\mathcal{V} \otimes \mathcal{W}^\vee$ and simply write $\text{deg}(\mathcal{V} \otimes \mathcal{W}^\vee)^{\geq 0}$.

We collect some basic facts about the slope theory for vector bundles on X .

Proposition 2.2.9. *The isomorphism class of every vector bundle \mathcal{V} on X is determined by the HN polygon $\text{HN}(\mathcal{V})$.*

Proof. Let us write the HN decomposition of \mathcal{V} as

$$\mathcal{V} \simeq \bigoplus_{i=1}^l \mathcal{O}(\lambda_i)^{\oplus m_i}$$

where $\lambda_1 > \lambda_2 > \dots > \lambda_l$. It suffices to show that we can find the numbers λ_i and m_i from $\text{HN}(\mathcal{V})$. Let x_i and y_i respectively denote the horizontal and vertical length of the i -th segment in $\text{HN}(\mathcal{V})$. From definition we find

$$\begin{aligned} x_i &= \text{rk}(\mathcal{V}^{\geq \lambda_i}) - \text{rk}(\mathcal{V}^{\geq \lambda_{i-1}}) = \text{rk}(\mathcal{O}(\lambda_i)^{\oplus m_i}) = m_i \cdot \text{rk}(\mathcal{O}(\lambda_i)), \\ y_i &= \text{deg}(\mathcal{V}^{\geq \lambda_i}) - \text{deg}(\mathcal{V}^{\geq \lambda_{i-1}}) = \text{deg}(\mathcal{O}(\lambda_i)^{\oplus m_i}) = m_i \cdot \text{deg}(\mathcal{O}(\lambda_i)). \end{aligned}$$

We thus find λ_i by

$$\lambda_i = \mu(\mathcal{O}(\lambda_i)) = \frac{\text{deg}(\mathcal{O}(\lambda_i))}{\text{rk}(\mathcal{O}(\lambda_i))} = \frac{y_i}{x_i}.$$

We then obtain $\text{rk}(\mathcal{O}(\lambda_i))$ and $\text{deg}(\mathcal{O}(\lambda_i))$ by Lemma 2.2.5, and in turn find m_i by

$$m_i = \frac{x_i}{\text{rk}(\mathcal{O}(\lambda_i))} = \frac{y_i}{\text{deg}(\mathcal{O}(\lambda_i))}.$$

□

Lemma 2.2.10. *Let \mathcal{V} be a vector bundle on X . We have identities*

$$\text{rk}(\mathcal{V}) = \text{rk}(\mathcal{V}^\vee) \quad \text{and} \quad \text{deg}(\mathcal{V}) = -\text{deg}(\mathcal{V}^\vee).$$

Moreover, for every $\mu \in \mathbb{Q}$ we have identities

$$\text{rk}(\mathcal{V}^{\geq \mu}) = \text{rk}((\mathcal{V}^\vee)^{\leq -\mu}) \quad \text{and} \quad \text{deg}(\mathcal{V}^{\geq \mu}) = -\text{deg}((\mathcal{V}^\vee)^{\leq -\mu}).$$

Proof. We verify the first statement for stable \mathcal{V} by Lemma 2.2.5, then extend it to general \mathcal{V} using the HN decomposition. We then deduce the second statement from the first statement by observing $(\mathcal{V}^{\geq \mu})^\vee \simeq (\mathcal{V}^\vee)^{\leq -\mu}$ using Lemma 2.2.5. □

Lemma 2.2.11. *Given two vector bundles \mathcal{V} and \mathcal{W} on X with $\mu_{\min}(\mathcal{V}) > \mu_{\max}(\mathcal{W})$, we have*

$$\text{Hom}(\mathcal{V}, \mathcal{W}) = 0.$$

Proof. Using the HN decomposition, we immediately reduce to the case when both \mathcal{V} and \mathcal{W} are stable. Note that the condition $\mu_{\min}(\mathcal{V}) > \mu_{\max}(\mathcal{W})$ now becomes $\mu(\mathcal{V}) > \mu(\mathcal{W})$.

Suppose for contradiction that there exists a nonzero bundle map $f : \mathcal{V} \rightarrow \mathcal{W}$. Let \mathcal{Q} denote the image of this map, which is nonzero by our assumption. Since \mathcal{Q} is a subbundle of \mathcal{W} , stability of \mathcal{W} yields

$$\mu(\mathcal{Q}) \leq \mu(\mathcal{W}). \tag{2.3}$$

On the other hand, the surjective bundle map $\mathcal{V} \rightarrow \mathcal{Q}$ gives an injective dual map $\mathcal{Q}^\vee \hookrightarrow \mathcal{V}^\vee$. Since stability of \mathcal{V} implies stability of \mathcal{V}^\vee by Lemma 2.2.5, we obtain an inequality

$$\mu(\mathcal{Q}^\vee) \leq \mu(\mathcal{V}^\vee).$$

By Lemma 2.2.10, this inequality is equivalent to

$$\mu(\mathcal{V}) \leq \mu(\mathcal{Q}). \tag{2.4}$$

Now we combine (2.3) and (2.4) to find $\mu(\mathcal{V}) \leq \mu(\mathcal{W})$, thereby completing the proof by contradiction. □

2.3. Moduli of bundle maps.

In this subsection we define certain moduli spaces of bundle maps over X and discuss some of their key properties. The reader can find a detailed discussion about these spaces in [BFH⁺17, §3.3].

Let us first define these moduli spaces as functors on the category of perfectoid spaces over $\mathrm{Spa}(F)$, which we denote by $\mathrm{Perf}/_{\mathrm{Spa}(F)}$. Note that, by construction, the relative Fargues-Fontaine curve X_S for any $S \in \mathrm{Spa}(F)$ comes with a natural map $X_S \rightarrow X$.

Definition 2.3.1. Let \mathcal{E} and \mathcal{F} be vector bundles on X . For any $S \in \mathrm{Spa}(F)$, we denote by \mathcal{E}_S and \mathcal{F}_S the vector bundles on X_S obtained as the pullback of \mathcal{E} and \mathcal{F} along the map $X_S \rightarrow X$.

- (1) $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ is the functor associating to every $S \in \mathrm{Perf}/_{\mathrm{Spa}(F)}$ the set of \mathcal{O}_{X_S} -module maps $m : \mathcal{E}_S \rightarrow \mathcal{F}_S$.
- (2) $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ is the functor associating to every $S \in \mathrm{Perf}/_{\mathrm{Spa}(F)}$ the set of surjective \mathcal{O}_{X_S} -module maps $m : \mathcal{E}_S \twoheadrightarrow \mathcal{F}_S$.
- (3) $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})$ is the functor associating to every $S \in \mathrm{Perf}/_{\mathrm{Spa}(F)}$ the set of \mathcal{O}_{X_S} -module maps $m : \mathcal{E}_S \rightarrow \mathcal{F}_S$ whose pullback along the map $X_{\bar{x}} \rightarrow X_S$ for any geometric point $\bar{x} \rightarrow S$ gives an injective $\mathcal{O}_{X_{\bar{x}}}$ -module map.

It turns out that we can make sense of these functors as moduli spaces in the category of diamonds as defined by Scholze [Sch18].

Proposition 2.3.2 ([BFH⁺17, Proposition 3.3.2, Proposition 3.3.5 and Proposition 3.3.6]). *Let \mathcal{E} and \mathcal{F} be vector bundles on X . The functors $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$, $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ and $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})$ are all locally spatial and partially proper diamonds in the sense of Scholze [Sch18]. Moreover, we have the following facts:*

- (1) *The diamond $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ is equidimensional of dimension $\deg(\mathcal{E}^\vee \otimes \mathcal{F}) \geq 0$.*
- (2) *Every nonempty open subfunctor of $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ has an F -point.*
- (3) *The diamonds $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ and $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})$ are both open subfunctors of $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$.*

Remark. The functor $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ is also a Banach-Colmez space as defined by Colmez [Col02]. Moreover, its dimension as a diamond coincides with its “principal” dimension as a Banach-Colmez space.

Definition 2.3.3. We write $|\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})|$, $|\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})|$ and $|\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})|$ respectively for the underlying topological space of the diamonds $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$, $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ and $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})$.

For the proof of our main theorem, we will consider a stratification of the $\mathcal{H}\mathrm{om}$ space according to the isomorphism type of image.

Definition 2.3.4. Given vector bundles \mathcal{E}, \mathcal{F} and \mathcal{Q} on X , we define $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}$ as the image of the map of diamonds

$$\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{Q}) \times_{\mathrm{Spd} F} \mathcal{I}\mathrm{nj}(\mathcal{Q}, \mathcal{F}) \rightarrow \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$$

induced by composition of bundle maps, and denote by $|\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}|$ the underlying topological space of $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}$.

Proposition 2.3.5 ([BFH⁺17, Proposition 3.3.9 and Lemma 3.3.10]). *Given vector bundles \mathcal{E}, \mathcal{F} and \mathcal{Q} on X , the topological space $|\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}|$ satisfies the following properties:*

- (1) *$|\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}|$ is stable under generalization and specialization inside $|\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})|$.*
- (2) *If $|\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}|$ is nonempty, its dimension is given by*

$$\dim |\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}| = \deg(\mathcal{E}^\vee \otimes \mathcal{Q}) \geq 0 + \deg(\mathcal{Q}^\vee \otimes \mathcal{F}) \geq 0 - \deg(\mathcal{Q}^\vee \otimes \mathcal{Q}) \geq 0.$$

3. CLASSIFICATION OF SUBBUNDLES

3.1. The main theorem and primary reductions.

The rest of this paper will be devoted to establishing our main result as stated below, where by a subbundle we mean a locally free subsheaf.

Theorem 3.1.1. *Let \mathcal{F} be a vector bundle on X . Then a vector bundle \mathcal{E} on X is a subbundle of \mathcal{F} if and only if the following equivalent conditions are satisfied:*

- (i) $\text{rk}(\mathcal{E}^{\geq \mu}) \leq \text{rk}(\mathcal{F}^{\geq \mu})$ for every $\mu \in \mathbb{Q}$.
- (ii) For each $i = 1, \dots, \text{rank}(\mathcal{E})$, the slope of $\text{HN}(\mathcal{E})$ on the interval $[i-1, i]$ is less than or equal to the slope of $\text{HN}(\mathcal{F})$ on this interval.

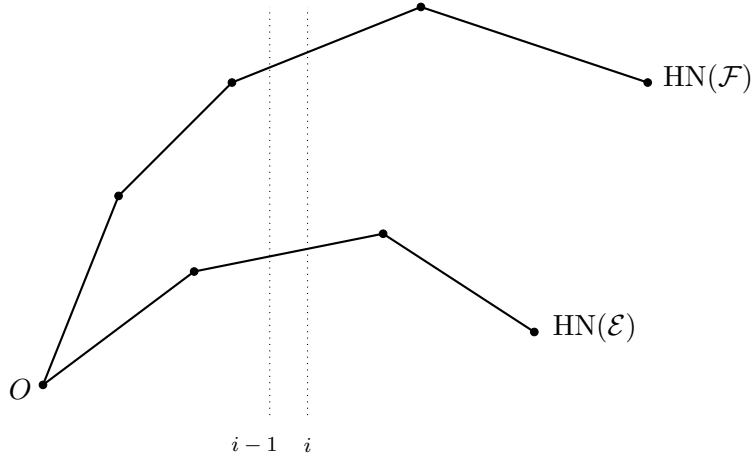


FIGURE 2. Illustration of the condition (ii) in Theorem 3.1.1.

We begin our proof of Theorem 3.1.1 by proving necessity of the condition (i).

Proposition 3.1.2. *Given a vector bundle \mathcal{F} on X , every subbundle \mathcal{E} of \mathcal{F} satisfies the condition (i) in Theorem 3.1.1.*

Proof. Let μ be an arbitrary rational number, and choose an injective bundle map $\mathcal{E} \hookrightarrow \mathcal{F}$. Lemma 2.2.11 implies that this map should embed $\mathcal{E}^{\geq \mu}$ into $\mathcal{F}^{\geq \mu}$, thereby yielding the desired inequality $\text{rk}(\mathcal{E}^{\geq \mu}) \leq \text{rk}(\mathcal{F}^{\geq \mu})$. \square

For sufficiency of the condition (i), we note the following easy but important reduction.

Proposition 3.1.3. *It suffices to prove sufficiency of the condition (i) in Theorem 3.1.1 under the assumption that \mathcal{E} and \mathcal{F} have no common slopes.*

Proof. Let \mathcal{E} and \mathcal{F} be vector bundles on X which satisfy the condition (i) in Theorem 3.1.1. From HN decompositions, we find decompositions

$$\mathcal{E} \simeq \mathcal{U} \oplus \acute{\mathcal{E}} \quad \text{and} \quad \mathcal{F} \simeq \mathcal{U} \oplus \acute{\mathcal{F}} \quad (3.1)$$

where $\acute{\mathcal{E}}$ and $\acute{\mathcal{F}}$ have no common slopes. For every $\mu \in \mathbb{Q}$ the decompositions (3.1) yield

$$\text{rk}(\mathcal{E}^{\geq \mu}) = \text{rk}(\mathcal{U}^{\geq \mu}) + \text{rk}(\acute{\mathcal{E}}^{\geq \mu}) \quad \text{and} \quad \text{rk}(\mathcal{F}^{\geq \mu}) = \text{rk}(\mathcal{U}^{\geq \mu}) + \text{rk}(\acute{\mathcal{F}}^{\geq \mu}).$$

Since \mathcal{E} and \mathcal{F} satisfy the condition (i) in Theorem 3.1.1, we consequently find

$$\text{rk}(\acute{\mathcal{E}}^{\geq \mu}) \leq \text{rk}(\acute{\mathcal{F}}^{\geq \mu}) \quad \text{for every } \mu \in \mathbb{Q}.$$

Moreover, an injective map $\acute{\mathcal{E}} \hookrightarrow \acute{\mathcal{F}}$ gives rise to an injective map $\mathcal{E} \hookrightarrow \mathcal{F}$ by direct summing with the identity map on \mathcal{U} . Hence we may prove sufficiency of the condition (i) in Theorem 3.1.1 after replacing \mathcal{E} and \mathcal{F} by $\acute{\mathcal{E}}$ and $\acute{\mathcal{F}}$. We thus have the desired reduction as $\acute{\mathcal{E}}$ and $\acute{\mathcal{F}}$ have no common slopes. \square

We now consider equivalence of the two conditions in Theorem 3.1.1. For convenience, we define the condition (ii) in Theorem 3.1.1 as a separate notion.

Definition 3.1.4. Let \mathcal{E} and \mathcal{F} be vector bundles on X . We say that \mathcal{F} *slopeswise dominates* \mathcal{E} if the condition (ii) in Theorem 3.1.1 is satisfied.

This notion is originally introduced by the author in [Hon19] where equivalence of the conditions (i) and (ii) in Theorem 3.1.1 is proved in the following form:

Proposition 3.1.5 ([Hon19, Lemma 4.2.2]). *Let \mathcal{E} and \mathcal{F} be vector bundles on X . Then we have $\text{rk}(\mathcal{E}^{\geq \mu}) \leq \text{rk}(\mathcal{F}^{\geq \mu})$ for every $\mu \in \mathbb{Q}$ if and only if \mathcal{F} slopeswise dominates \mathcal{E} .*

This equivalence will be extremely useful to us since the notion of slopeswise dominance has several implications that are not easy to deduce directly from its equivalent condition (i) in Theorem 3.1.1.

Lemma 3.1.6 ([Hon19, Lemma 4.2.3, Lemma 4.2.4, and Lemma 4.2.5]). *Let \mathcal{E} and \mathcal{F} be vector bundles on X such that \mathcal{F} slopeswise dominates \mathcal{E} .*

(1) *We have an inequality*

$$\deg(\mathcal{F})^{\geq 0} \geq \deg(\mathcal{E})^{\geq 0}.$$

(2) *There exist decompositions*

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{E}' \quad \text{and} \quad \mathcal{F} \simeq \mathcal{D} \oplus \mathcal{F}' \tag{3.2}$$

satisfying the following properties:

(i) \mathcal{F}' slopeswise dominates \mathcal{E}' .

(ii) If $\mathcal{E}' \neq 0$, we have $\mu_{\max}(\mathcal{F}') > \mu_{\max}(\mathcal{E}')$.

(iii) If $\mathcal{D} \neq 0$ and $\mathcal{E}' \neq 0$, we have $\mu_{\min}(\mathcal{D}) \geq \mu_{\max}(\mathcal{F}') > \mu_{\max}(\mathcal{E}')$.

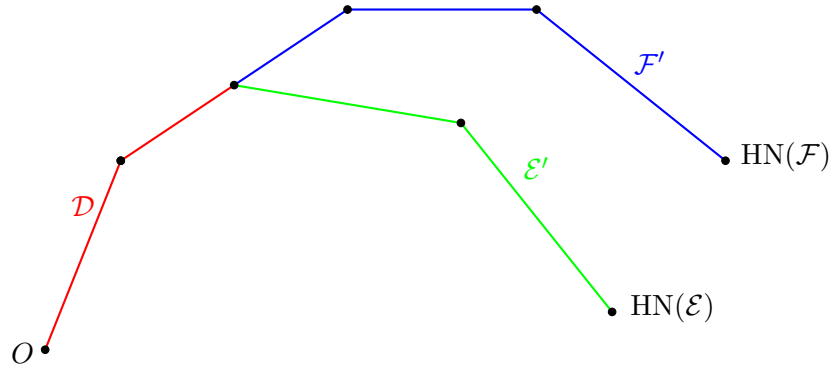


FIGURE 3. Illustration of the decompositions (3.2) in terms of HN polygons.

(3) *If $\text{rk}(\mathcal{E}) = \text{rk}(\mathcal{F})$, then \mathcal{E}^\vee slopeswise dominates \mathcal{F}^\vee .*

Remark. The proof of [Hon19, Lemma 4.2.4] shows that the bundle \mathcal{D} in (3.2) represents the common part of $\text{HN}(\mathcal{E})$ and $\text{HN}(\mathcal{F})$, as illustrated in Figure 3.

3.2. Formulation of the key inequality.

Thus far, by Propositions 3.1.2, 3.1.3 and 3.1.5 we have reduced the proof of Theorem 3.1.1 to establishing sufficiency of the condition (ii) under the additional assumption that \mathcal{E} and \mathcal{F} have no common slopes. In this subsection, we will further reduce it to establishing the following quantitative statement:

Proposition 3.2.1. *Let \mathcal{E} , \mathcal{F} and \mathcal{Q} be vector bundles on X with the following properties:*

- (i) \mathcal{F} slopewise dominates \mathcal{E} .
- (ii) \mathcal{E}^\vee slopewise dominates \mathcal{Q}^\vee .
- (iii) \mathcal{F} slopewise dominates \mathcal{Q} .
- (iv) \mathcal{E} and \mathcal{F} have no common slopes.
- (v) $\text{rk}(\mathcal{Q}) < \text{rk}(\mathcal{E})$.

Then we have an inequality

$$\deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} < \deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0}. \quad (3.3)$$

Remark. This is the analogue of [Hon19, Proposition 4.3.5] in our situation. In fact, the statement of Proposition 3.2.1 and the statement of [Hon19, Proposition 4.3.5] have several notable similarities as follows:

- (1) The inequalities considered in both statements are almost identical.
- (2) By Proposition 3.1.5, the conditions (ii) and (iii) in Proposition 3.2.1 are almost equivalent to the corresponding conditions (ii) and (iii) in [Hon19, Proposition 4.3.5]
- (3) Both statements have the condition (iv) in common.
- (4) The condition (i) in each statement is precisely the condition on \mathcal{E} and \mathcal{F} in the corresponding main theorem in each context.

On the other hand, the statement of Proposition 3.2.1 contains some changes from the statement of [Hon19, Proposition 4.3.5] as follows:

- (1) The condition (ii) in Proposition 3.2.1 does not have an additional “equality condition” that appears in the condition (ii) in [Hon19, Proposition 4.3.5].
- (2) Proposition 3.2.1 has an additional condition (v) which is not present in [Hon19, Proposition 4.3.5].
- (3) The inequality (3.3) in Proposition 3.2.1 is strict whereas the inequality considered in [Hon19, Proposition 4.3.5] is not.

Here the essential feature is the additional condition (v), as the other two features are consequences of this feature.

It is relatively easy to see why the strictness of (3.3) is a consequence of the additional condition (v). In fact, if we remove the condition (v) from Proposition 3.2.1, then both sides of (3.3) can be equal for many choices of \mathcal{E} , \mathcal{F} and \mathcal{Q} . As an example, the reader can quickly check that both sides of (3.3) are equal whenever $\mathcal{E} = \mathcal{Q}$. There are also other choices, such as

$$\mathcal{E} = \mathcal{O}, \quad \mathcal{F} = \mathcal{O}(1) \oplus \mathcal{O}(-1), \quad \mathcal{Q} = \mathcal{O}(1),$$

for which both sides of (3.3) are equal.

Let us now explain why the absence of the “equality condition” in the condition (ii) is a consequence of the condition (v). As Lemma 3.2.3 indicates, for our purpose we only need to establish the inequality (3.3) when \mathcal{Q} is a quotient of \mathcal{E} and a subbundle of \mathcal{F} . The “equality condition” in the condition (ii) of [Hon19, Proposition 4.3.5] is a necessary condition for \mathcal{Q}

to be a quotient of \mathcal{E} . In our context, the condition (v) replaces this equality condition as a necessary condition for \mathcal{Q} to be a quotient of \mathcal{E} with $\mathcal{Q} \neq \mathcal{E}$. In fact, if we added the “equality condition” to the condition (ii) in Proposition 3.2.1, the case $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{E})$ would degenerate to the case $\mathcal{Q} = \mathcal{E}$.

Our discussion in the previous two paragraphs suggests that it is possible to state Proposition 3.2.1 without having all these new features by just adding the “equality condition” to the condition (ii). However, we still want to have these features, as these features will notably simplify a number of our reduction arguments in the proof of Proposition 3.2.1.

For the desired reduction, we need the following dual counterpart of Proposition 3.1.2 and Proposition 3.1.5.

Proposition 3.2.2. *Let \mathcal{E} be a vector bundle on X , and let \mathcal{Q} be a quotient bundle of \mathcal{E} . Then \mathcal{E}^\vee slopewise dominates \mathcal{Q}^\vee .*

Proof. Since \mathcal{Q} is a quotient bundle of \mathcal{E} , its dual bundle \mathcal{Q}^\vee is a subbundle of \mathcal{E}^\vee . We thus have slopewise dominance of \mathcal{E}^\vee on \mathcal{Q}^\vee by Proposition 3.1.2 and Proposition 3.1.5. \square

The following lemma relates some conditions in Proposition 3.2.1 to the construction that we introduced in Definition 2.3.4.

Lemma 3.2.3. *Let \mathcal{E}, \mathcal{F} and \mathcal{Q} be vector bundles on X such that the diamond $\text{Hom}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}$ (defined in Definition 2.3.4) is nonempty. Then the following statements hold:*

- (1) \mathcal{Q} is a quotient bundle of \mathcal{E} and a subbundle of \mathcal{F} .
- (2) The bundles \mathcal{E}, \mathcal{F} and \mathcal{Q} satisfy the conditions (ii) and (iii) in Proposition 3.2.1.
- (3) If $\mathcal{Q} \neq \mathcal{E}$, then $\text{rk}(\mathcal{Q}) < \text{rk}(\mathcal{E})$.

Proof. By definition, $\text{Hom}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}$ is the image of the map of diamonds

$$\text{Surj}(\mathcal{E}, \mathcal{Q}) \times_{\text{Spd } F} \text{Inj}(\mathcal{Q}, \mathcal{F}) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{F})$$

induced by composition of bundle maps. Nonemptiness of $\text{Hom}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}$ therefore implies that both $\text{Surj}(\mathcal{E}, \mathcal{F})$ and $\text{Inj}(\mathcal{E}, \mathcal{F})$ are nonempty. By Proposition 2.3.2, we find that both $\text{Surj}(\mathcal{E}, \mathcal{Q})$ and $\text{Inj}(\mathcal{E}, \mathcal{Q})$ have F -points, which precisely amounts to existence of a surjective bundle map $\mathcal{E} \twoheadrightarrow \mathcal{Q}$ and an injective bundle map $\mathcal{Q} \hookrightarrow \mathcal{F}$ as asserted in (1). Furthermore, we deduce (2) from (1) by Propositions 3.1.2, 3.1.5, and 3.2.2.

Let us now assume that $\mathcal{Q} \neq \mathcal{E}$. As we already saw in the preceding paragraph, there exists a surjective map $\mathcal{E} \twoheadrightarrow \mathcal{Q}$. Its kernel \mathcal{K} is not trivial since the map is not an isomorphism by our assumption. We thus find

$$\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{E}) - \text{rk}(\mathcal{K}) < \text{rk}(\mathcal{E}),$$

thereby establishing (3). \square

With Lemma 3.2.3, we can explain why establishing Proposition 3.2.1 finishes the proof of Theorem 3.1.1.

Proposition 3.2.4. *Proposition 3.2.1 implies sufficiency of the condition (ii) in Theorem 3.1.1 under the additional assumption that \mathcal{E} and \mathcal{F} have no common slopes.*

Proof. Let \mathcal{E} and \mathcal{F} be vector bundles on X with no common slopes such that \mathcal{F} slopewise dominates \mathcal{E} . Let S be the set of (isomorphism classes of) vector bundles \mathcal{Q} on X such that $\text{Hom}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}$ is nonempty. We wish to prove that \mathcal{E} is a subbundle of \mathcal{F} , assuming Proposition 3.2.1. By Lemma 3.2.3, it is enough to show $\mathcal{E} \in S$.

Suppose for contradiction that $\mathcal{E} \notin S$. By Proposition 3.2.1 and Lemma 3.2.3, every $\mathcal{Q} \in S$ should satisfy the strict inequality

$$\deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} < \deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0}.$$

Now the dimension formulas in Proposition 2.3.2 and Proposition 2.3.5 imply that for every $\mathcal{Q} \in S$ we have

$$\dim |\mathcal{H}om(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}| < \dim \mathcal{H}om(\mathcal{E}, \mathcal{F}). \quad (3.4)$$

On the other hand, we have a decomposition

$$|\mathcal{H}om(\mathcal{E}, \mathcal{F})| = \coprod_{\mathcal{Q} \in S} |\mathcal{H}om(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}|.$$

We thus use Proposition 2.3.5 and (3.4) to find

$$\dim |\mathcal{H}om(\mathcal{E}, \mathcal{F})| = \sup_{\mathcal{Q} \in S} \dim |\mathcal{H}om(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}| < \dim |\mathcal{H}om(\mathcal{E}, \mathcal{F})|,$$

thereby obtaining the desired contradiction. \square

3.3. Reduction on slopes and ranks.

Our goal for the rest of this paper is to establish Proposition 3.2.1. For our convenience, we introduce the following notation:

Definition 3.3.1. For arbitrary vector bundles \mathcal{E}, \mathcal{F} and \mathcal{Q} on X , we define

$$c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q}) := \deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0} - \deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} - \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0}.$$

Note that the inequality (3.3) in Proposition 3.2.1 can be written as $c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q}) > 0$.

In this subsection, we reduce the proof of Proposition 3.2.1 to the case where the following additional conditions are satisfied:

- (v)' $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{E}) - 1$.
- (vi) all slopes of \mathcal{E}, \mathcal{F} and \mathcal{Q} are integers.
- (vii) $\mu_{\max}(\mathcal{E}) = 0$.

The following lemma will be crucial for this task.

Lemma 3.3.2 ([Hon19, Lemma 3.2.7 and Lemma 3.2.8]). *Let \mathcal{V} and \mathcal{W} be arbitrary vector bundles on X .*

- (1) *For vector bundles $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$ on X whose HN polygons are obtained by vertically stretching $\text{HN}(\mathcal{V})$ and $\text{HN}(\mathcal{W})$ by a positive integer factor C , we have*

$$\deg(\tilde{\mathcal{V}}^\vee \otimes \tilde{\mathcal{W}})^{\geq 0} = C \cdot \deg(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0}.$$

- (2) *For vector bundles $\mathcal{V}(\lambda) := \mathcal{V} \otimes \mathcal{O}(\lambda)$ and $\mathcal{W}(\lambda) := \mathcal{W} \otimes \mathcal{O}(\lambda)$, we have*

$$\deg(\mathcal{V}(\lambda)^\vee \otimes \mathcal{W}(\lambda))^{\geq 0} = \text{rk}(\mathcal{O}(\lambda))^2 \cdot \deg(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0}.$$

Let us now carry out the proposed reduction.

Proposition 3.3.3. *It suffices to prove Proposition 3.2.1 under the assumption that all slopes of \mathcal{E}, \mathcal{F} and \mathcal{Q} are integers.*

Proof. Let \mathcal{E}, \mathcal{F} and \mathcal{Q} be as in the statement of Proposition 3.2.1. Take C to be a common multiple of all denominators of the slopes in $\text{HN}(\mathcal{E}), \text{HN}(\mathcal{F})$ and $\text{HN}(\mathcal{Q})$, and define $\tilde{\mathcal{E}}, \tilde{\mathcal{F}}$ and $\tilde{\mathcal{Q}}$ to be vector bundles on X whose HN polygons are obtained by vertically stretching $\text{HN}(\mathcal{E}), \text{HN}(\mathcal{F})$ and $\text{HN}(\mathcal{Q})$ by a factor C . Then we have the following facts:

- (1) All slopes of $\tilde{\mathcal{E}}, \tilde{\mathcal{F}}$ and $\tilde{\mathcal{Q}}$ are integers.
- (2) The conditions (i) - (iv) in Proposition 3.2.1 are satisfied after replacing \mathcal{E}, \mathcal{F} and \mathcal{Q} by $\tilde{\mathcal{E}}, \tilde{\mathcal{F}}$ and $\tilde{\mathcal{Q}}$.
- (3) $\text{rk}(\mathcal{Q}) = \text{rk}(\tilde{\mathcal{Q}})$ and $\text{rk}(\mathcal{E}) = \text{rk}(\tilde{\mathcal{E}})$.
- (4) $c_{\tilde{\mathcal{E}}, \tilde{\mathcal{F}}}(\tilde{\mathcal{Q}}) = C \cdot c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q})$.

Indeed, (1), (2) and (3) are evident by construction while (4) follows from Lemma 3.3.2. Now (2), (3) and (4) together imply that we may prove Proposition 3.2.1 after replacing \mathcal{E}, \mathcal{F} and \mathcal{Q} by $\tilde{\mathcal{E}}, \tilde{\mathcal{F}}$ and $\tilde{\mathcal{Q}}$, thereby yielding the desired reduction by (1). \square

Proposition 3.3.4. *It suffices to prove Proposition 3.2.1 under the following additional conditions:*

- (v)' $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{E}) - 1$.
- (vi) *all slopes of \mathcal{E}, \mathcal{F} and \mathcal{Q} are integers.*

Proof. Suppose that Proposition 3.2.1 holds when the conditions (v)' and (vi) are satisfied. We wish to deduce the general case of Proposition 3.2.1 from this assumption. In light of Proposition 3.3.3, we assume that the condition (vi) is satisfied. Under this assumption, we proceed by induction on $\text{rk}(\mathcal{E}) - \text{rk}(\mathcal{Q})$. Since the base case $\text{rk}(\mathcal{E}) - \text{rk}(\mathcal{Q}) = 1$ follows from our assumption, we only need to consider the induction step.

We first reduce our induction step to the case $\mu_{\max}(\mathcal{E}) = 0$. For this, we take $\lambda := \mu_{\max}(\mathcal{E})$ and consider the vector bundles

$$\mathcal{E}(-\lambda) := \mathcal{E} \otimes \mathcal{O}(-\lambda), \quad \mathcal{F}(-\lambda) := \mathcal{F} \otimes \mathcal{O}(-\lambda), \quad \mathcal{Q}(-\lambda) := \mathcal{Q} \otimes \mathcal{O}(-\lambda).$$

Note that $\lambda = \mu_{\max}(\mathcal{E})$ is an integer by the condition (vi) that we assumed. In particular, the bundle $\mathcal{O}(\lambda)$ has rank 1 by Lemma 2.2.5. It is therefore straightforward to check the following identity using Definition 2.2.4.

$$\mathcal{O}(\mu) \otimes \mathcal{O}(-\lambda) = \mathcal{O}(\mu - \lambda) \quad \text{for all } \mu \in \mathbb{Q}.$$

Then by HN decompositions we observe that $\text{HN}(\mathcal{E}(-\lambda)), \text{HN}(\mathcal{F}(-\lambda))$ and $\text{HN}(\mathcal{Q}(-\lambda))$ are obtained by reducing all slopes of $\text{HN}(\mathcal{E}), \text{HN}(\mathcal{F})$ and $\text{HN}(\mathcal{Q})$ by λ . Consequently, we deduce the following facts:

- (1) $\mu_{\max}(\mathcal{E}(-\lambda)) = \mu_{\max}(\mathcal{E}) - \lambda = 0$.
- (2) The conditions (i) - (iv) in Proposition 3.2.1 and the additional condition (vi) are satisfied after replacing \mathcal{E}, \mathcal{F} and \mathcal{Q} by $\mathcal{E}(-\lambda), \mathcal{F}(-\lambda)$ and $\mathcal{Q}(-\lambda)$.
- (3) $\text{rk}(\mathcal{Q}(-\lambda)) = \text{rk}(\mathcal{Q})$ and $\text{rk}(\mathcal{E}(-\lambda)) = \text{rk}(\mathcal{E})$.

Moreover, by Lemma 3.3.2 we get an identity

$$c_{\mathcal{E}(-\lambda), \mathcal{F}(-\lambda)}(\mathcal{Q}(-\lambda)) = c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q}) \tag{3.5}$$

since $\text{rk}(\mathcal{O}(-\lambda)) = 1$ as already noted. Now (2), (3) and (3.5) together imply that we may replace \mathcal{E}, \mathcal{F} and \mathcal{Q} by $\mathcal{E}(-\lambda), \mathcal{F}(-\lambda)$ and $\mathcal{Q}(-\lambda)$ for the induction step, thereby yielding the desired reduction by (1).

Let us now assume that $\mu_{\max}(\mathcal{E}) = 0$. For our induction step we assume $\text{rk}(\mathcal{E}) - \text{rk}(\mathcal{Q}) > 1$, or equivalently $\text{rk}(\mathcal{E}) > \text{rk}(\mathcal{Q}) + 1$. Then we can write

$$\mathcal{E} = \check{\mathcal{E}} \oplus \mathcal{O} \tag{3.6}$$

where $\mu_{\max}(\check{\mathcal{E}}) \leq 0$ and $\text{rk}(\check{\mathcal{E}}) > \text{rk}(\mathcal{Q})$.

Our next assertion is that the conditions (i) - (iv) in Proposition 3.2.1 and the additional condition (vi) are satisfied after replacing \mathcal{E} by $\check{\mathcal{E}}$. The condition (iii) is trivial since \mathcal{F} and \mathcal{Q}

remain unchanged. The condition (iv) and the additional condition (vi) are also obvious by construction. For the condition (i), we need to check slopewise dominance of \mathcal{F} on $\check{\mathcal{E}}$, which follows by combining slopewise dominance of \mathcal{F} on \mathcal{E} and slopewise dominance of \mathcal{E} on $\check{\mathcal{E}}$; in fact, the former is given by the condition (i) for \mathcal{E} and \mathcal{F} , whereas the latter follows by applying Proposition 3.1.2 and Proposition 3.1.5 to the observation that $\check{\mathcal{E}}$ is a subbundle of \mathcal{E} by (3.6). For the remaining condition (ii), we need to show slopewise dominance of $\check{\mathcal{E}}^\vee$ on \mathcal{Q}^\vee . From (3.6) we obtain

$$\mathcal{E}^\vee = \mathcal{O} \oplus \check{\mathcal{E}}^\vee. \quad (3.7)$$

Moreover, by Lemma 2.2.5, we have $\mu_{\min}(\mathcal{E}^\vee) = -\mu_{\max}(\mathcal{E}) = 0$ and $\mu_{\min}(\check{\mathcal{E}}^\vee) = -\mu_{\max}(\check{\mathcal{E}})$. Hence we see that $\text{HN}(\check{\mathcal{E}}^\vee)$ is obtained from $\text{HN}(\mathcal{E}^\vee)$ by removing the line segment over the interval $(\text{rk}(\mathcal{E}) - 1, \text{rk}(\mathcal{E})]$, as indicated in Figure 4. Since $\text{rk}(\mathcal{E}) > \text{rk}(\mathcal{Q})$ by our assumption, this removal process does not affect slopewise dominance on \mathcal{Q}^\vee . In other words, slopewise dominance of \mathcal{E}^\vee on \mathcal{Q}^\vee as given in the condition (ii) implies slopewise dominance of $\check{\mathcal{E}}^\vee$ on \mathcal{Q}^\vee as desired.

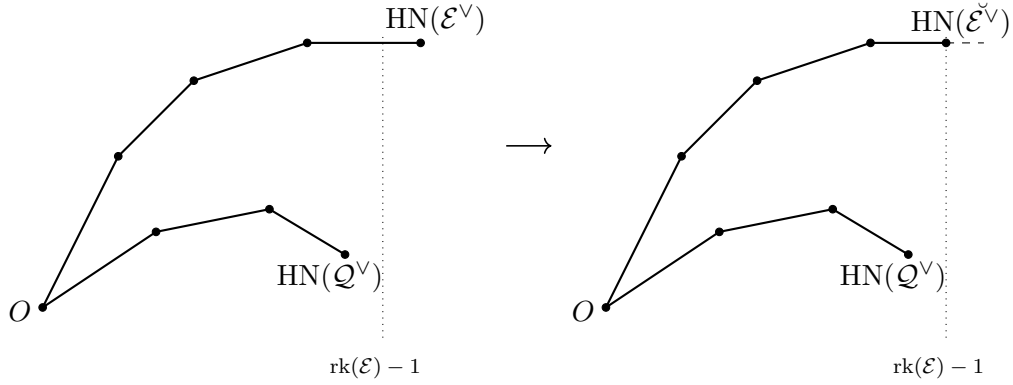


FIGURE 4. Illustration of the induction step in terms of dual HN polygons

Now, as $\text{rk}(\check{\mathcal{E}}) - \text{rk}(\mathcal{Q}) < \text{rk}(\mathcal{E}) - \text{rk}(\mathcal{Q})$, our discussion in the preceding paragraph shows that we may apply the induction hypothesis to deduce

$$c_{\check{\mathcal{E}}, \mathcal{F}}(\mathcal{Q}) > 0 \quad (3.8)$$

For the desired inequality $c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q}) > 0$, we use the decomposition (3.7) to compute

$$\begin{aligned} \deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} &= \deg((\check{\mathcal{E}}^\vee \oplus \mathcal{O}) \otimes \mathcal{F})^{\geq 0} \\ &= \deg(\check{\mathcal{E}}^\vee \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{O} \otimes \mathcal{F})^{\geq 0} \\ &= \deg(\check{\mathcal{E}}^\vee \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{F})^{\geq 0}, \\ \deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} &= \deg((\check{\mathcal{E}}^\vee \oplus \mathcal{O}) \otimes \mathcal{Q})^{\geq 0} \\ &= \deg(\check{\mathcal{E}}^\vee \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{O} \otimes \mathcal{Q})^{\geq 0} \\ &= \deg(\check{\mathcal{E}}^\vee \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{Q})^{\geq 0}. \end{aligned}$$

Then by Definition 3.3.1 we find

$$c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q}) = c_{\check{\mathcal{E}}, \mathcal{F}}(\mathcal{Q}) + \deg(\mathcal{F})^{\geq 0} - \deg(\mathcal{Q})^{\geq 0}.$$

Since \mathcal{F} slopewise dominates \mathcal{Q} by the condition (iii), we use Lemma 3.1.6 to find

$$c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q}) \geq c_{\check{\mathcal{E}}, \mathcal{F}}(\mathcal{Q}). \quad (3.9)$$

We thus deduce the desired inequality $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) > 0$ from (3.8) and (3.9). \square

Remark. Proposition 3.3.3 and Proposition 3.3.4 are the counterparts of [Hon19, Proposition 4.4.5 and Proposition 4.4.6] in our setting. Naturally, their proofs closely follow the proofs of their counterparts.

Here we note a notable difference between Proposition 3.3.4 and its counterpart [Hon19, Proposition 4.4.6]. In Proposition 3.3.4, our reduction does not reach the case $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{E})$; on the other hand, the reduction in [Hon19, Proposition 4.4.6] reaches the case $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{F})$. We will see that our argument in §3.4 requires some additional work because of this difference.

At first glance, this difference seems to be a direct consequence of the condition (v) in Proposition 3.2.1. However, even if we remove this condition from Proposition 3.2.1, we are still unable to reach the case $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{E})$ by our reduction argument in Proposition 3.3.4. The main issue is that, as remarked in §3.2, removing the condition (v) from Proposition 3.2.1 makes the inequality (3.3) a nonstrict inequality where equality may hold even if $\mathcal{Q} \not\cong \mathcal{E}$. In fact, in the proof of [Hon19, Proposition 4.4.6] the reduction to the case $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{F})$ crucially uses the equality condition $\mathcal{Q} \simeq \mathcal{F}$ for the inequality in [Hon19, Proposition 4.3.5].

We also point out that our proof of Proposition 3.3.3 and Proposition 3.3.4 enjoys the benefits from several features of Proposition 3.2.1 as remarked in §3.2. For example, when we replace the triple $(\mathcal{E}, \mathcal{F}, \mathcal{Q})$ by another triple, such as $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}, \tilde{\mathcal{Q}})$ in the proof of Proposition 3.3.3 or $(\mathcal{E}(-\lambda), \mathcal{F}(-\lambda), \mathcal{Q}(-\lambda))$ in the proof of Proposition 3.3.4, it is straightforward to check the condition (ii) in Proposition 3.2.1 for the new triple because of the absence of the equality condition.

Proposition 3.3.5. *We may prove Proposition 3.2.1 under the following additional conditions:*

- (v)' $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{E}) - 1$.
- (vi) all slopes of \mathcal{E}, \mathcal{F} and \mathcal{Q} are integers.
- (vii) $\mu_{\max}(\mathcal{E}) = 0$.

Proof. Let \mathcal{E}, \mathcal{F} and \mathcal{Q} be vector bundles on X which satisfy the conditions (v)' and (vi) in addition to all conditions in Proposition 3.2.1. By Proposition 3.3.4, it suffices to consider such vector bundles for the proof of Proposition 3.2.1. For the desired reduction, we can argue exactly as in the second paragraph of the proof of Proposition 3.3.4; in other words, we set $\lambda := \mu_{\max}(\mathcal{E})$ and replace \mathcal{E}, \mathcal{F} and \mathcal{Q} by

$$\mathcal{E}(-\lambda) := \mathcal{E} \otimes \mathcal{O}(-\lambda), \quad \mathcal{F}(-\lambda) := \mathcal{F} \otimes \mathcal{O}(-\lambda), \quad \mathcal{Q}(-\lambda) := \mathcal{Q} \otimes \mathcal{O}(-\lambda)$$

to obtain the desired reduction. \square

3.4. Degeneration of the dual bundles.

By Proposition 3.3.5, our remaining goal is to prove the following statement:

Proposition 3.4.1. *Let \mathcal{E}, \mathcal{F} and \mathcal{Q} be vector bundles on X with the following properties:*

- (i) \mathcal{F} slopewise dominates \mathcal{E} .
- (ii) \mathcal{E}^\vee slopewise dominates \mathcal{Q}^\vee .
- (iii) \mathcal{F} slopewise dominates \mathcal{Q} .
- (iv) \mathcal{E} and \mathcal{F} have no common slopes.
- (v) $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{E}) - 1$.
- (vi) all slopes of \mathcal{E}, \mathcal{F} and \mathcal{Q} are integers.
- (vii) $\mu_{\max}(\mathcal{E}) = 0$.

Then we have an inequality

$$c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) > 0. \quad (3.10)$$

For the rest of this paper, we fix vector bundles \mathcal{E}, \mathcal{F} and \mathcal{Q} as in the statement of Proposition 3.4.1.

Let us briefly sketch our proof of Proposition 3.4.1. The key idea is to construct a finite sequence

$$\mathcal{E} = \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_r = \mathcal{Q}$$

which is “dually degenerating” in the sense that \mathcal{E}_i^\vee slopewise dominates \mathcal{E}_{i+1}^\vee for each $i = 0, 1, \dots, r$. By this “degenerating” property, we will obtain

$$c_{\mathcal{E}_i,\mathcal{F}}(\mathcal{Q}) \geq c_{\mathcal{E}_{i+1},\mathcal{F}}(\mathcal{Q}) \quad \text{for each } i = 0, 1, \dots, r-1. \quad (3.11)$$

Consequently we will deduce

$$c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) = c_{\mathcal{E}_0,\mathcal{F}}(\mathcal{Q}) \geq c_{\mathcal{E}_r,\mathcal{F}}(\mathcal{Q}) = c_{\mathcal{Q},\mathcal{F}}(\mathcal{Q}) = 0 \quad (3.12)$$

where the last identity follows immediately from Definition 3.3.1. We will then show that equality in (3.12) never holds by examining the equality condition of the inequality (3.4.9).

Remark. Our proof of Proposition 3.4.1 will closely follow the argument in [Hon19, §4.4]. However, there are some adjustments that we need to make.

In [Hon19, §4.4], the construction of the degenerating sequence crucially relies on the condition $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{F})$. In our context, since we begin with the condition $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{E}) - 1$, we will need an additional step to attain a similar “equal rank” condition. We will thus construct \mathcal{E}_1 by cutting down \mathcal{E} so that we have $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{E}_1)$.

The main subtlety for our proof of Proposition 3.4.1 lies in establishing nonstrictness of the inequality (3.10). In [Hon19, §4.4], the equality condition for the inequality $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) \geq 0$ is established by showing that $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q})$ strictly decreases during the first step, or more precisely $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) > c_{\mathcal{E},\mathcal{F}_1}(\mathcal{Q})$. In our situation, we will have to simultaneously consider the first two steps because of the additional step that we described in the preceding paragraph. Our argument also requires some additional adjustments on details as we will see in the proof of Proposition 3.4.10.

We now begin our proof of Proposition 3.4.1. As remarked above, the first step of our construction aims to attain an equal rank condition by cutting down \mathcal{E} .

Proposition 3.4.2. *Let \mathcal{E}_1 be a direct summand of \mathcal{E} such that*

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{O}.$$

Then we have the following facts:

- (1) \mathcal{E}_1 and \mathcal{F} have no common slopes.
- (2) $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{E}_1)$
- (3) all slopes of \mathcal{E}_1 are integers.
- (4) $\mu_{\max}(\mathcal{E}_1) \leq 0$.
- (5) \mathcal{Q} slopewise dominates \mathcal{E}_1
- (6) We have an inequality

$$c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) \geq c_{\mathcal{E}_1,\mathcal{F}}(\mathcal{Q})$$

with equality if and only if $\deg(\mathcal{F})^{\geq 0} = \deg(\mathcal{Q})^{\geq 0}$.

Proof. By construction, the statements (1), (2), (3) and (4) follow immediately from the conditions (iv), (v), (vi) and (vii) in Proposition 3.4.1. In addition, the condition (ii) and the condition (vii) in Proposition 3.4.1 together yield slopewise dominance of \mathcal{E}_1^\vee on \mathcal{Q}^\vee , which consequently implies the statement (5) by Lemma 3.1.6 and the statement (2). Moreover, we can argue as in the last paragraph of the proof of Proposition 3.3.4 to find

$$c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q}) = c_{\mathcal{E}_1, \mathcal{F}}(\mathcal{Q}) + \deg(\mathcal{F})^{\geq 0} - \deg(\mathcal{Q})^{\geq 0},$$

from which the statement (6) follows by Lemma 3.1.6 and the condition (iii) in Proposition 3.4.1. \square

In order to describe the rest of our construction, we recall the following notion from [Hon19]:

Definition 3.4.3. Let \mathcal{V} and \mathcal{W} be nonzero vector bundles on X with integer slopes such that \mathcal{V} slopewise dominates \mathcal{W} . We refer to the vector bundle

$$\bar{\mathcal{V}} := \mathcal{O}(\mu_{\max}(\mathcal{W}))^{\oplus \text{rk}(\mathcal{V}^{>\mu_{\max}(\mathcal{W})})} \oplus \mathcal{V}^{\leq \mu_{\max}(\mathcal{W})}$$

as the *maximal slope reduction* of \mathcal{V} to \mathcal{W} . In other words, $\bar{\mathcal{V}}$ is the vector bundle on X obtained from \mathcal{V} by reducing all slopes of $\mathcal{V}^{>\mu_{\max}(\mathcal{W})}$ to $\mu_{\max}(\mathcal{W})$.

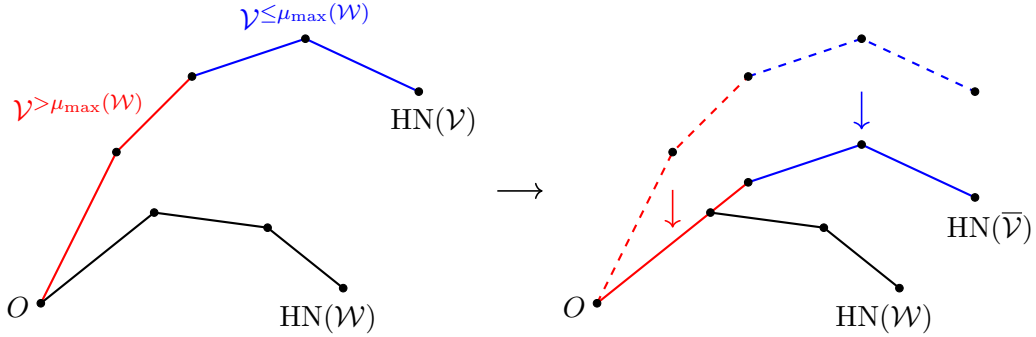


FIGURE 5. Illustration of the maximal slope reduction

We note some basic properties of the maximal slope reduction.

Lemma 3.4.4. Let \mathcal{V} and \mathcal{W} be nonzero vector bundles on X with integer slopes such that \mathcal{V} slopewise dominates \mathcal{W} . Let $\bar{\mathcal{V}}$ denote the maximal slope reduction of \mathcal{V} to \mathcal{W} . Then we have the following facts:

- (1) $\mu_{\max}(\bar{\mathcal{V}}) = \mu_{\max}(\mathcal{W})$.
- (2) $\text{rk}(\bar{\mathcal{V}}) = \text{rk}(\mathcal{V})$.
- (3) $\mathcal{V} = \bar{\mathcal{V}}$ if and only if $\mu_{\max}(\mathcal{V}) = \mu_{\max}(\mathcal{W})$.
- (4) $\bar{\mathcal{V}}$ slopewise dominates \mathcal{W} .
- (5) all slopes of $\bar{\mathcal{V}}$ are integers.

Proof. All statements follow immediately from Definition 3.4.3. \square

We also note a computational lemma that we will use.

Lemma 3.4.5 ([BFH⁺17, Lemma 2.3.4]). *Let \mathcal{V} and \mathcal{W} be any vector bundles on X with HN decompositions*

$$\mathcal{V} \simeq \bigoplus_{i=1}^p \mathcal{O}(\lambda_i)^{\oplus m_i} \quad \text{and} \quad \mathcal{W} \simeq \bigoplus_{j=1}^q \mathcal{O}(\kappa_j)^{\oplus n_j}$$

where $\lambda_1 > \lambda_2 > \dots > \lambda_p$ and $\kappa_1 > \kappa_2 > \dots > \kappa_q$. We represent the i -th segment in $\text{HN}(\mathcal{V})$ and j -th segment in $\text{HN}(\mathcal{W})$ (from left to right) by the vectors v_i and w_j , respectively. More precisely, we set

$$v_i := (\text{rk}(\mathcal{O}(\lambda_i)^{\oplus m_i}), \text{deg}(\mathcal{O}(\lambda_i)^{\oplus m_i})) \quad \text{and} \quad w_j := (\text{rk}(\mathcal{O}(\kappa_j)^{\oplus n_j}), \text{deg}(\mathcal{O}(\kappa_j)^{\oplus n_j})).$$

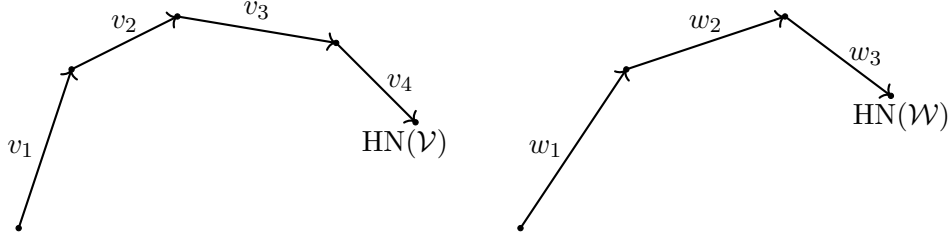


FIGURE 6. Vector representation of HN polygons.

If we write $\mu(v_i)$ and $\mu(w_j)$ respectively for the slopes of v_i and w_j , we have an identity

$$\text{deg}(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0} = \sum_{\mu(v_i) \leq \mu(w_j)} v_i \times w_j$$

where $v_i \times w_j$ denotes the two-dimensional cross product of the vectors v_i and w_j . In particular, we have $\text{deg}(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0} = 0$ if $\mu_{\min}(\mathcal{V}) \geq \mu_{\max}(\mathcal{W})$.

Remark. Given two vectors $v = (a, b)$ and $w = (c, d)$ in \mathbb{R}^2 , we define their two-dimensional cross product by $v \times w := ad - bc$. If v and w are regarded as vectors in \mathbb{R}^3 with zero z -coordinates, then our definition recovers the z -coordinate of their conventional cross product.

Let us now proceed to the inductive part of our construction.

Proposition 3.4.6. *We can construct a sequence of vector bundles $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \dots$ so that the following statements hold for each $i = 1, 2, \dots$.*

(1) *There exist decompositions*

$$\mathcal{Q}^\vee \simeq \mathcal{M}_i \oplus \mathcal{R}_i \quad \text{and} \quad \mathcal{E}_i^\vee \simeq \mathcal{M}_i \oplus \mathcal{S}_i$$

which satisfy the following properties:

(a) \mathcal{S}_i slopewise dominates \mathcal{R}_i .

(b) If $\mathcal{S}_i \neq 0$, we have $\mu_{\max}(\mathcal{S}_i) > \mu_{\max}(\mathcal{R}_i)$.

(c) If $\mathcal{M}_i \neq 0$ and $\mathcal{S}_i \neq 0$, we have $\mu_{\min}(\mathcal{M}_i) \geq \mu_{\max}(\mathcal{S}_i) > \mu_{\max}(\mathcal{R}_i)$.

(2) *If $i > 1$ we have*

$$\mathcal{E}_i \simeq \begin{cases} \mathcal{Q} & \text{if } \mathcal{E}_{i-1} \simeq \mathcal{Q} \\ \mathcal{M}_{i-1}^\vee \oplus \bar{\mathcal{S}}_{i-1}^\vee & \text{otherwise} \end{cases}$$

where $\bar{\mathcal{S}}_{i-1}$ denotes the maximal slope reduction of \mathcal{S}_{i-1} to \mathcal{R}_{i-1} .

(3) $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{E}_i)$.

(4) all slopes of \mathcal{E}_i are integers.

(5) \mathcal{Q} slopewise dominates \mathcal{E}_i .

Proof. Let us take \mathcal{E}_1 as in Proposition 3.4.2. We deduce the statements (3), (4) and (5) for $i = 1$ directly follows from Proposition 3.4.2. Moreover, we obtain the statement (1) for $i = 1$ as a formal consequence of the statements (3) and (5) by Lemma 3.1.6. Since the statement (2) for $i = 1$ vacuously holds, we have thus verified all statements for $i = 1$.

We now proceed by induction on i . If $\mathcal{E}_i \simeq \mathcal{Q}$, the induction step becomes trivial; in fact, if we take $\mathcal{E}_{i+1} := \mathcal{Q}$, the statement (2) for $i + 1$ is obvious by construction while the other statements (1), (3), (4) and (5) for $i + 1$ immediately follow from the induction hypothesis as $\mathcal{E}_{i+1} \simeq \mathcal{E}_i$. We thus assume from now on that $\mathcal{E}_i \not\simeq \mathcal{Q}$. By the induction hypothesis, the statement (1) for i yields decompositions

$$\mathcal{Q}^\vee \simeq \mathcal{M}_i \oplus \mathcal{R}_i \quad \text{and} \quad \mathcal{E}_i^\vee \simeq \mathcal{M}_i \oplus \mathcal{S}_i \quad (3.13)$$

where \mathcal{R}_i slopewise dominates \mathcal{S}_i . Moreover, the statement (4) for i and the condition (vi) in Proposition 3.4.1 together imply that all slopes of \mathcal{R}_i and \mathcal{S}_i are integers. Hence it makes sense to consider the maximal slope reduction of \mathcal{S}_i to \mathcal{R}_i , which we denote by $\overline{\mathcal{S}}_i$. Let us now take

$$\mathcal{E}_{i+1} := \mathcal{M}_i^\vee \oplus \overline{\mathcal{S}}_i^\vee. \quad (3.14)$$

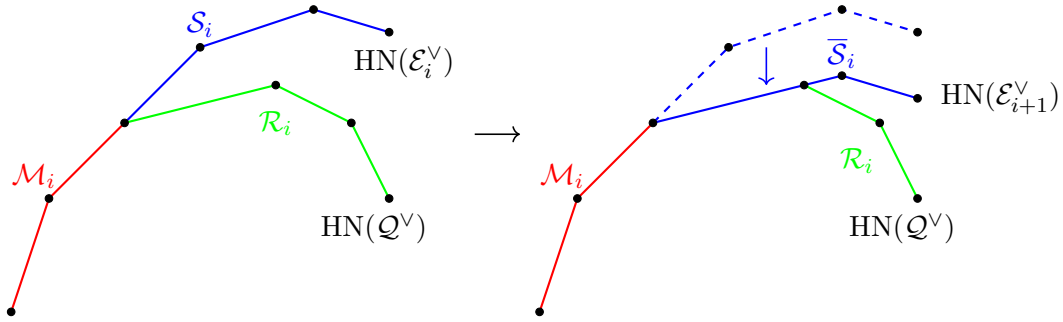


FIGURE 7. Construction of the sequence (\mathcal{E}_i)

The statement (2) for $i + 1$ is obvious by our definition of \mathcal{E}_{i+1} in (3.14). We also verify the statement (3) for $i + 1$ by computing

$$\text{rk}(\mathcal{E}_{i+1}) = \text{rk}(\mathcal{M}_i) + \text{rk}(\overline{\mathcal{S}}_i) = \text{rk}(\mathcal{M}_i) + \text{rk}(\mathcal{S}_i) = \text{rank}(\mathcal{E}_i) = \text{rk}(\mathcal{Q}) \quad (3.15)$$

where for each equality we use (3.14), Lemma 3.4.4, (3.13) and the statement (3) for i . Moreover, since all slopes of \mathcal{M}_i and \mathcal{S}_i are integers by the statement (4) for i , we obtain the statement (4) for $i + 1$ using (3.14) and Lemma 3.4.4. Furthermore, since $\overline{\mathcal{S}}_i$ slopewise dominates \mathcal{R}_i by Lemma 3.4.4, we deduce slopewise dominance of \mathcal{E}_{i+1}^\vee on \mathcal{Q}^\vee from decompositions

$$\mathcal{E}_{i+1}^\vee = \mathcal{M}_i \oplus \overline{\mathcal{S}}_i \quad \text{and} \quad \mathcal{Q}^\vee = \mathcal{M}_i \oplus \mathcal{R}_i$$

as given by (3.13) and (3.14), and consequently verify the statement (1) for $i + 1$ by Lemma 3.1.6. In addition, slopewise dominance of \mathcal{E}_{i+1}^\vee on \mathcal{Q}^\vee implies the statement (5) for $i + 1$ by Lemma 3.1.6 and (3.15). We thus have all statements in Proposition 3.4.6 for $i + 1$, thereby concluding our proof by induction. \square

Remark. From our construction it is not hard to see that \mathcal{E}_{i+1}^\vee slopewise dominates \mathcal{E}_i^\vee for all i , as we proposed while sketching our proof of Proposition 3.4.1. Although this “dually degenerating” property won’t explicitly appear in our argument, it will play a crucial role in the proof of Proposition 3.4.9 under the guise of relations between slopes of \mathcal{S}_i and $\overline{\mathcal{S}}_i$.

We record a couple of simple but useful observations about the construction in Proposition 3.4.6.

Lemma 3.4.7. *Let $\mathcal{E}_1, \mathcal{E}_2, \dots$ be as in Proposition 3.4.6. For each $i = 1, 2, \dots$ we take decompositions*

$$\mathcal{Q}^\vee \simeq \mathcal{M}_i \oplus \mathcal{R}_i \quad \text{and} \quad \mathcal{E}_i^\vee \simeq \mathcal{M}_i \oplus \mathcal{S}_i$$

as given by the statement (1) in Proposition 3.4.6.

- (1) The bundle \mathcal{M}_i represents the common part of $\text{HN}(\mathcal{Q}^\vee)$ and $\text{HN}(\mathcal{E}_i^\vee)$.
- (2) If $\mathcal{E}_i \not\simeq \mathcal{Q}$, we have $\mathcal{R}_i \neq 0$ and $\mathcal{S}_i \neq 0$.

Proof. The first statement follows from the remark after Lemma 3.1.6. The second statement is an immediate consequence of the first statement. \square

Our construction process turns out to be essentially finite in the sense that the sequence stabilizes after finitely many steps, as the following proposition shows.

Proposition 3.4.8. *Let $\mathcal{E}_1, \mathcal{E}_2, \dots$ be as in Proposition 3.4.6. There exists $r > 0$ such that $\mathcal{E}_i \simeq \mathcal{Q}$ for all $i \geq r$.*

Proof. By the statement (2) in Proposition 3.4.6, it suffices to show that $\mathcal{E}_i \simeq \mathcal{Q}$ for some i . Suppose for contradiction that $\mathcal{E}_i \not\simeq \mathcal{Q}$ for all i . Let us take decompositions

$$\mathcal{Q}^\vee \simeq \mathcal{M}_i \oplus \mathcal{R}_i \quad \text{and} \quad \mathcal{E}_i^\vee \simeq \mathcal{M}_i \oplus \mathcal{S}_i \tag{3.16}$$

as given by the statement (1) in Proposition 3.4.6. As $\mathcal{E}_i \not\simeq \mathcal{Q}$ by our assumption, the statement (2) in Proposition 3.4.6 yields a decomposition

$$\mathcal{E}_{i+1}^\vee \simeq \mathcal{M}_i \oplus \overline{\mathcal{S}}_i \tag{3.17}$$

where $\overline{\mathcal{S}}_i$ denotes the maximal slope reduction of \mathcal{S}_i to \mathcal{R}_i . Since \mathcal{R}_i and \mathcal{S}_i are both nonzero by Lemma 3.4.7, the polygons $\text{HN}(\mathcal{R}_i)$ and $\text{HN}(\overline{\mathcal{S}}_i)$ must have a nontrivial common part, which we represent by a nonzero vector bundle \mathcal{T}_i on X . Then we deduce from the decompositions (3.16) and (3.17) that the common part of $\text{HN}(\mathcal{Q}^\vee)$ and $\text{HN}(\mathcal{E}_{i+1}^\vee)$ should include $\text{HN}(\mathcal{M}_i \oplus \mathcal{T}_i)$. Now by Lemma 3.4.7 we find

$$\text{rk}(\mathcal{M}_{i+1}) \geq \text{rk}(\mathcal{M}_i \oplus \mathcal{T}_i) = \text{rk}(\mathcal{M}_i) + \text{rk}(\mathcal{T}_i) > \text{rk}(\mathcal{M}_i).$$

In particular, the sequence $(\text{rk}(\mathcal{M}_i))$ should be unbounded. However, this is impossible since we have $\text{rk}(\mathcal{M}_i) \leq \text{rk}(\mathcal{Q})$ by (3.16). We thus complete the proof by contradiction. \square

We now prove the essential property of our sequence.

Proposition 3.4.9. *Let $\mathcal{E}_1, \mathcal{E}_2, \dots$ be as in Proposition 3.4.6. For each $i = 1, 2, \dots$, we take decompositions*

$$\mathcal{Q}^\vee \simeq \mathcal{M}_i \oplus \mathcal{R}_i \quad \text{and} \quad \mathcal{E}_i^\vee \simeq \mathcal{M}_i \oplus \mathcal{S}_i \tag{3.18}$$

as given by the statement (1) in Proposition 3.4.6. Then for each $i = 1, 2, \dots$, we have an inequality

$$c_{\mathcal{E}_i, \mathcal{F}}(\mathcal{Q}) \geq c_{\mathcal{E}_{i+1}, \mathcal{F}}(\mathcal{Q}) \tag{3.19}$$

where equality holds only if $\mathcal{E}_i \simeq \mathcal{Q}$ or $\text{rk}(\mathcal{S}_i^\vee) = \text{rk}(\mathcal{F}^{>\mu_{\min}(\mathcal{S}_i^\vee)})$.

Proof. If $\mathcal{E}_i \simeq \mathcal{Q}$, the proof is trivial as we have $\mathcal{E}_{i+1} \simeq \mathcal{Q} \simeq \mathcal{E}_i$ by the statement (2) in Proposition 3.4.6. We thus assume from now on that $\mathcal{E}_i \not\simeq \mathcal{Q}$.

Let $\overline{\mathcal{S}}_i$ denote the maximal slope reduction of \mathcal{S}_i to \mathcal{R}_i . Then we have a decomposition

$$\mathcal{E}_{i+1}^\vee \simeq \mathcal{M}_i \oplus \overline{\mathcal{S}}_i \quad (3.20)$$

by the statement (2) in Proposition 3.4.6. Moreover, since $\mathcal{E}_i \not\simeq \mathcal{Q}$ we have $\mathcal{R}_i \neq 0$ and $\mathcal{S}_i \neq 0$ by Lemma 3.4.7. Hence the statement (1) in Proposition 3.4.6 yields

$$\mu_{\min}(\mathcal{M}_i) \geq \mu_{\max}(\mathcal{S}_i) > \mu_{\max}(\mathcal{R}_i) = \mu_{\max}(\overline{\mathcal{S}}_i) \quad \text{if } \mathcal{M}_i \neq 0. \quad (3.21)$$

Then by Lemma 3.4.5 we obtain

$$\deg(\mathcal{M}_i^\vee \otimes \mathcal{S}_i)^{\geq 0} = \deg(\mathcal{M}_i^\vee \otimes \overline{\mathcal{S}}_i)^{\geq 0} = 0. \quad (3.22)$$

Now we use (3.18), (3.20) and (3.22) to find

$$\begin{aligned} \deg(\mathcal{E}_i^\vee \otimes \mathcal{F})^{\geq 0} &= \deg((\mathcal{M}_i \oplus \mathcal{S}_i) \otimes \mathcal{F})^{\geq 0} = \deg(\mathcal{M}_i \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{S}_i \otimes \mathcal{F})^{\geq 0}, \\ \deg(\mathcal{E}_{i+1}^\vee \otimes \mathcal{F})^{\geq 0} &= \deg((\mathcal{M}_i \oplus \overline{\mathcal{S}}_i) \otimes \mathcal{F})^{\geq 0} = \deg(\mathcal{M}_i \otimes \mathcal{F})^{\geq 0} + \deg(\overline{\mathcal{S}}_i \otimes \mathcal{F})^{\geq 0}, \\ \deg(\mathcal{E}_i^\vee \otimes \mathcal{Q})^{\geq 0} &= \deg((\mathcal{M}_i \oplus \mathcal{S}_i) \otimes (\mathcal{M}_i^\vee \oplus \mathcal{R}_i^\vee))^{\geq 0} \\ &= \deg(\mathcal{M}_i \otimes \mathcal{M}_i^\vee)^{\geq 0} + \deg(\mathcal{M}_i \otimes \mathcal{R}_i^\vee)^{\geq 0} + \deg(\mathcal{S}_i \otimes \mathcal{M}_i^\vee)^{\geq 0} + \deg(\mathcal{S}_i \otimes \mathcal{R}_i^\vee)^{\geq 0} \\ &= \deg(\mathcal{M}_i \otimes \mathcal{M}_i^\vee)^{\geq 0} + \deg(\mathcal{M}_i \otimes \mathcal{R}_i^\vee)^{\geq 0} + \deg(\mathcal{S}_i \otimes \mathcal{R}_i^\vee)^{\geq 0}, \\ \deg(\mathcal{E}_{i+1}^\vee \otimes \mathcal{Q})^{\geq 0} &= \deg((\mathcal{M}_i \oplus \overline{\mathcal{S}}_i) \otimes (\mathcal{M}_i^\vee \oplus \mathcal{R}_i^\vee))^{\geq 0} \\ &= \deg(\mathcal{M}_i \otimes \mathcal{M}_i^\vee)^{\geq 0} + \deg(\mathcal{M}_i \otimes \mathcal{R}_i^\vee)^{\geq 0} + \deg(\overline{\mathcal{S}}_i \otimes \mathcal{M}_i^\vee)^{\geq 0} + \deg(\overline{\mathcal{S}}_i \otimes \mathcal{R}_i^\vee)^{\geq 0} \\ &= \deg(\mathcal{M}_i \otimes \mathcal{M}_i^\vee)^{\geq 0} + \deg(\mathcal{M}_i \otimes \mathcal{R}_i^\vee)^{\geq 0} + \deg(\overline{\mathcal{S}}_i \otimes \mathcal{R}_i^\vee)^{\geq 0}. \end{aligned}$$

Therefore by Definition 3.3.1 we have

$$\begin{aligned} c_{\mathcal{E}_i, \mathcal{F}}(\mathcal{Q}) - c_{\mathcal{E}_{i+1}, \mathcal{F}}(\mathcal{Q}) &= (\deg(\mathcal{E}_i^\vee \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{E}_{i+1}^\vee \otimes \mathcal{F})^{\geq 0}) - (\deg(\mathcal{E}_i^\vee \otimes \mathcal{Q})^{\geq 0} - \deg(\mathcal{E}_{i+1}^\vee \otimes \mathcal{Q})^{\geq 0}) \\ &= (\deg(\mathcal{S}_i \otimes \mathcal{F})^{\geq 0} - \deg(\overline{\mathcal{S}}_i \otimes \mathcal{F})^{\geq 0}) - (\deg(\mathcal{S}_i \otimes \mathcal{R}_i^\vee)^{\geq 0} - \deg(\overline{\mathcal{S}}_i \otimes \mathcal{R}_i^\vee)^{\geq 0}) \end{aligned}$$

The desired inequality (3.19) is thus equivalent to

$$\deg(\mathcal{S}_i \otimes \mathcal{F})^{\geq 0} - \deg(\overline{\mathcal{S}}_i \otimes \mathcal{F})^{\geq 0} \geq \deg(\mathcal{S}_i \otimes \mathcal{R}_i^\vee)^{\geq 0} - \deg(\overline{\mathcal{S}}_i \otimes \mathcal{R}_i^\vee)^{\geq 0}. \quad (3.23)$$

Let us set $\lambda := \mu_{\max}(\mathcal{R}_i)$ and $r := \text{rk}(\mathcal{S}_i^{>\lambda})$. Since $\overline{\mathcal{S}}_i$ is the maximal slope reduction of \mathcal{S}_i , we have decompositions

$$\mathcal{S}_i \simeq \mathcal{S}_i^{>\lambda} \oplus \mathcal{S}_i^{\leq\lambda} \quad \text{and} \quad \overline{\mathcal{S}}_i \simeq \mathcal{O}(\lambda)^{\oplus r} \oplus \mathcal{S}_i^{\leq\lambda}.$$

Then we have

$$\begin{aligned} \deg(\mathcal{S}_i \otimes \mathcal{F})^{\geq 0} &= \deg((\mathcal{S}_i^{>\lambda} \oplus \mathcal{S}_i^{\leq\lambda}) \otimes \mathcal{F})^{\geq 0} \\ &= \deg(\mathcal{S}_i^{>\lambda} \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{S}_i^{\leq\lambda} \otimes \mathcal{F})^{\geq 0}, \\ \deg(\overline{\mathcal{S}}_i \otimes \mathcal{F})^{\geq 0} &= \deg((\mathcal{O}(\lambda)^{\oplus r} \oplus \mathcal{S}_i^{\leq\lambda}) \otimes \mathcal{F})^{\geq 0} \\ &= \deg(\mathcal{O}(\lambda)^{\oplus r} \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{S}_i^{\leq\lambda} \otimes \mathcal{F})^{\geq 0}, \\ \deg(\mathcal{S}_i \otimes \mathcal{R}_i^\vee)^{\geq 0} &= \deg((\mathcal{S}_i^{>\lambda} \oplus \mathcal{S}_i^{\leq\lambda}) \otimes \mathcal{R}_i^\vee)^{\geq 0} \\ &= \deg(\mathcal{S}_i^{>\lambda} \otimes \mathcal{R}_i^\vee)^{\geq 0} + \deg(\mathcal{S}_i^{\leq\lambda} \otimes \mathcal{R}_i^\vee)^{\geq 0}, \\ \deg(\overline{\mathcal{S}}_i \otimes \mathcal{R}_i^\vee)^{\geq 0} &= \deg((\mathcal{O}(\lambda)^{\oplus r} \oplus \mathcal{S}_i^{\leq\lambda}) \otimes \mathcal{R}_i^\vee)^{\geq 0} \\ &= \deg(\mathcal{O}(\lambda)^{\oplus r} \otimes \mathcal{R}_i^\vee)^{\geq 0} + \deg(\mathcal{S}_i^{\leq\lambda} \otimes \mathcal{R}_i^\vee)^{\geq 0}. \end{aligned}$$

We can thus rewrite the inequality (3.23) as

$$\deg(\mathcal{S}_i^{>\lambda} \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{O}(\lambda)^{\oplus r} \otimes \mathcal{F})^{\geq 0} \geq \deg(\mathcal{S}_i^{>\lambda} \otimes \mathcal{R}_i^{\vee})^{\geq 0} - \deg(\mathcal{O}(\lambda)^{\oplus r} \otimes \mathcal{R}_i^{\vee})^{\geq 0}. \quad (3.24)$$

Let us now take sequences of vectors $(r_a), (s_b)$ and (f_c) which respectively represent the line segments in $\text{HN}(\mathcal{R}_i), \text{HN}(\mathcal{S}_i^{>\lambda})$ and $\text{HN}(\mathcal{F}^{\vee})$. Let us also set $s := \sum s_b$ and take \bar{s} to be a vector representing the only line segment in $\text{HN}(\mathcal{O}(\lambda)^{\oplus r})$. By construction, we obtain

$$\begin{aligned} s &= (\text{rk}(\mathcal{S}_i^{>\lambda}), \deg(\mathcal{S}_i^{>\lambda})) = (\text{rk}(\mathcal{S}_i^{>\lambda}), \mu(\mathcal{S}_i^{>\lambda}) \cdot \text{rk}(\mathcal{S}_i^{>\lambda})) = (r, r\mu(\mathcal{S}_i^{>\lambda})), \\ \bar{s} &= (\text{rk}(\mathcal{O}(\lambda)^{\oplus r}), \deg(\mathcal{O}(\lambda)^{\oplus r})) = (\text{rk}(\mathcal{O}(\lambda)^{\oplus r}), \mu(\mathcal{O}(\lambda)^{\oplus r}) \cdot \text{rk}(\mathcal{O}(\lambda)^{\oplus r})) = (r, r\lambda). \end{aligned}$$

where we use the fact that $\lambda = \mu_{\max}(\mathcal{R}_i)$ is an integer by the decomposition (3.18) and the condition (vi) in Proposition 3.4.1. We thus have

$$s - \bar{s} = (0, r(\mu(\mathcal{S}_i^{>\lambda}) - \lambda)). \quad (3.25)$$

We now aim to estimate the left side of (3.24). By Lemma 3.4.5, we may write

$$\begin{aligned} \deg(\mathcal{S}_i^{>\lambda} \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{O}(\lambda)^{\oplus r} \otimes \mathcal{F})^{\geq 0} &= \deg(\mathcal{S}_i^{>\lambda} \otimes (\mathcal{F}^{\vee})^{\vee})^{\geq 0} - \deg(\mathcal{O}(\lambda)^{\oplus r} \otimes (\mathcal{F}^{\vee})^{\vee})^{\geq 0} \\ &= \sum_{\mu(f_c) \leq \mu(s_a)} f_c \times s_a - \sum_{\mu(f_c) \leq \lambda} f_c \times \bar{s}. \end{aligned} \quad (3.26)$$

Note that each s_a satisfies $\mu(s_a) > \lambda$ by construction. Hence each f_c with $\mu(f_c) \leq \lambda$ must satisfy $\mu(f_c) \leq \mu(s_a)$ for all s_a 's. We thereby obtain an inequality

$$\sum_{\mu(f_c) \leq \lambda} f_c \times s_a \leq \sum_{\mu(f_c) \leq \mu(s_a)} f_c \times s_a \quad (3.27)$$

as every term on each side is nonnegative. Now (3.26) yields

$$\begin{aligned} \deg(\mathcal{S}_i^{>\lambda} \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{O}(\lambda)^{\oplus r} \otimes \mathcal{F})^{\geq 0} &\geq \sum_{\mu(f_c) \leq \lambda} f_c \times s_a - \sum_{\mu(f_c) \leq \lambda} f_c \times \bar{s} \\ &= \sum_{\mu(f_c) \leq \lambda} f_c \times \sum s_a - \sum_{\mu(f_c) \leq \lambda} f_c \times \bar{s} \\ &= \sum_{\mu(f_c) \leq \lambda} f_c \times (s - \bar{s}). \end{aligned} \quad (3.28)$$

Moreover, as the sequence (f_c) represents the line segments in $\text{HN}(\mathcal{F}^{\vee})$ we have

$$\sum_{\mu(f_c) \leq \lambda} f_c = (\text{rk}((\mathcal{F}^{\vee})^{\leq \lambda}), \deg((\mathcal{F}^{\vee})^{\leq \lambda})),$$

and consequently obtain

$$\sum_{\mu(f_c) \leq \lambda} f_c \times (s - \bar{s}) = r \cdot \text{rk}((\mathcal{F}^{\vee})^{\leq \lambda}) \cdot (\mu(\mathcal{S}_i^{>\lambda}) - \lambda)$$

by (3.25). We can thus rewrite (3.28) as

$$\deg(\mathcal{S}_i^{>\lambda} \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{O}(\lambda)^{\oplus r} \otimes \mathcal{F})^{\geq 0} \geq r \cdot \text{rk}((\mathcal{F}^{\vee})^{\leq \lambda}) \cdot (\mu(\mathcal{S}_i^{>\lambda}) - \lambda). \quad (3.29)$$

Our next task is to compute the right side of (3.24). By Lemma 3.4.5 we have

$$\deg(\mathcal{S}_i^{>\lambda} \otimes \mathcal{R}_i^{\vee})^{\geq 0} - \deg(\mathcal{O}(\lambda)^{\oplus r} \otimes \mathcal{R}_i^{\vee})^{\geq 0} = \sum_{\mu(r_b) \leq \mu(s_a)} r_b \times s_a - \sum_{\mu(r_b) \leq \lambda} r_b \times \bar{s}. \quad (3.30)$$

Since the sequences (s_a) and (r_b) respectively represent the line segments in $\text{HN}(\mathcal{S}_i^{>\lambda})$ and $\text{HN}(\mathcal{R}_i)$, we have

$$\mu(r_b) \leq \mu_{\max}(\mathcal{R}_i) = \lambda < \mu(s_a)$$

for all s_a 's and r_b 's. Hence we can simplify (3.30) as

$$\begin{aligned} \deg(\mathcal{S}_i^{>\lambda} \otimes \mathcal{R}_i^{\vee})^{\geq 0} - \deg(\mathcal{O}(\lambda)^{\oplus r} \otimes \mathcal{R}_i^{\vee})^{\geq 0} &= \sum r_b \times s_a - \sum r_b \times \bar{s} \\ &= \sum r_b \times \sum s_a - \sum r_b \times \bar{s} \\ &= \sum r_b \times (s - \bar{s}). \end{aligned} \quad (3.31)$$

Moreover, by construction we have

$$\sum r_b = (\text{rk}(\mathcal{R}_i), \deg(\mathcal{R}_i)),$$

and consequently obtain

$$\sum r_b \times (s - \bar{s}) = r \cdot \text{rk}(\mathcal{R}_i) \cdot (\mu(\mathcal{S}_i^{>\lambda}) - \lambda)$$

by (3.25). We can thus rewrite (3.31) as

$$\deg(\mathcal{S}_i^{>\lambda} \otimes \mathcal{R}_i^{\vee})^{\geq 0} - \deg(\mathcal{O}(\lambda)^{\oplus r} \otimes \mathcal{R}_i^{\vee})^{\geq 0} = r \cdot \text{rk}(\mathcal{R}_i) \cdot (\mu(\mathcal{S}_i^{>\lambda}) - \lambda). \quad (3.32)$$

Since $\lambda = \mu_{\max}(\mathcal{R}_i)$, we use (3.18) to find

$$\text{rk}(\mathcal{R}_i) = \text{rk}(\mathcal{R}_i^{\leq \lambda}) \leq \text{rk}((\mathcal{Q}^{\vee})^{\leq \lambda}).$$

In addition, by Lemma 2.2.10, Proposition 3.1.5 and the condition (iii) in Proposition 3.4.1 we find

$$\text{rk}((\mathcal{Q}^{\vee})^{\leq \lambda}) = \text{rk}(\mathcal{Q}^{\geq -\lambda}) \leq \text{rk}(\mathcal{F}^{\geq -\lambda}) = \text{rk}((\mathcal{F}^{\vee})^{\leq \lambda}).$$

Hence we have

$$\text{rk}(\mathcal{R}_i) \leq \text{rk}((\mathcal{F}^{\vee})^{\leq \lambda}).$$

Furthermore, since $r = \text{rk}(\mathcal{S}_i^{>\lambda}) > 0$ by (3.21) and $\mu(\mathcal{S}_i^{>\lambda}) - \lambda > 0$ by definition, we obtain

$$r \cdot \text{rk}((\mathcal{F}^{\vee})^{\leq \lambda}) \cdot (\mu(\mathcal{S}_i^{>\lambda}) - \lambda) \geq r \cdot \text{rk}(\mathcal{R}_i) \cdot (\mu(\mathcal{S}_i^{>\lambda}) - \lambda). \quad (3.33)$$

Combining this with (3.29) and (3.32), we deduce the inequality (3.24) which is equivalent to the desired inequality (3.23).

Let us now consider the equality condition. From (3.33), we need

$$\text{rk}(\mathcal{R}_i) = \text{rk}((\mathcal{F}^{\vee})^{\leq \lambda}) \quad (3.34)$$

since both r and $\mu(\mathcal{S}_i^{>\lambda}) - \lambda$ are positive as already noted. We also need equality in (3.29), which requires equality in (3.27). Since every term on each side of (3.27) is nonnegative, we must have identical nonzero terms on both sides of (3.27). In particular, every f_c with $\mu(f_c) < \mu_{\max}(\mathcal{S}_i^{>\lambda})$ must satisfy $\mu(f_c) \leq \lambda$; indeed, for such an f_c we have a nonzero term $f_c \times s_a$ on the right side for some s_a with $\mu(s_a) = \mu_{\max}(\mathcal{S}_i^{>\lambda})$, and therefore must have the same nonzero term on the left side. We thus obtain

$$\text{rk}((\mathcal{F}^{\vee})^{\leq \lambda}) = \text{rk}((\mathcal{F}^{\vee})^{< \mu_{\max}(\mathcal{S}_i^{>\lambda})}) = \text{rk}((\mathcal{F}^{\vee})^{< \mu_{\max}(\mathcal{S}_i)}) \quad (3.35)$$

where for the second equality we observe $\mu_{\max}(\mathcal{S}_i^{>\lambda}) = \mu_{\max}(\mathcal{S}_i)$ by (3.21). Moreover, by Lemma 2.2.5 and Lemma 2.2.10 we have

$$\text{rk}((\mathcal{F}^{\vee})^{< \mu_{\max}(\mathcal{S}_i)}) = \text{rk}((\mathcal{F}^{\vee})^{< -\mu_{\min}(\mathcal{S}_i^{\vee})}) = \text{rk}(\mathcal{F}^{> \mu_{\min}(\mathcal{S}_i^{\vee})}). \quad (3.36)$$

We also have

$$\text{rk}(\mathcal{R}_i) = \text{rk}(\mathcal{S}_i) = \text{rk}(\mathcal{S}_i^{\vee}) \quad (3.37)$$

by (3.18), the statement (3) in Proposition 3.4.6 and Lemma 2.2.10. We then combine (3.34), (3.35), (3.36) and (3.37) to obtain an equality condition

$$\mathrm{rk}(\mathcal{S}_i^\vee) = \mathrm{rk}(\mathcal{F}^{>\mu_{\min}(\mathcal{S}_i^\vee)})$$

as desired. \square

Proposition 3.4.10. *Let $\mathcal{E}_1, \mathcal{E}_2, \dots$ be as in Proposition 3.4.6. Then we have a strict inequality*

$$c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q}) > c_{\mathcal{E}_2, \mathcal{F}}(\mathcal{Q}).$$

Proof. Proposition 3.4.2 and Proposition 3.4.9 together yield

$$c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q}) \geq c_{\mathcal{E}_1, \mathcal{F}}(\mathcal{Q}) \geq c_{\mathcal{E}_2, \mathcal{F}}(\mathcal{Q}). \quad (3.38)$$

We need to prove that at least one of the inequalities in (3.38) must be strict. We assume for contradiction that

$$c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q}) = c_{\mathcal{E}_1, \mathcal{F}}(\mathcal{Q}) = c_{\mathcal{E}_2, \mathcal{F}}(\mathcal{Q}). \quad (3.39)$$

Let us first consider the case $\mathcal{E}_1 \simeq \mathcal{Q}$. We note that

$$\mathrm{deg}(\mathcal{Q})^{\geq 0} = \mathrm{deg}(\mathcal{E}_1)^{\geq 0} = 0 \quad (3.40)$$

where the second equality follows from the statement (4) in Proposition 3.4.2. In addition, the first equality in (3.39) yields

$$\mathrm{deg}(\mathcal{F})^{\geq 0} = \mathrm{deg}(\mathcal{Q})^{\geq 0}. \quad (3.41)$$

by the statement (6) in Proposition 3.4.2. Now (3.40) and (3.41) together yield $\mathrm{deg}(\mathcal{F})^{\geq 0} = 0$, which in particular implies $\mu_{\max}(\mathcal{F}) \leq 0$. However, this is impossible because of the conditions (i), (iv) and (vii) in Proposition 3.4.1. We have thus obtained a desired contradiction.

Now it remains to consider the case $\mathcal{E}_1 \not\simeq \mathcal{Q}$. Let us take decompositions

$$\mathcal{Q}^\vee \simeq \mathcal{M}_1 \oplus \mathcal{R}_1 \quad \text{and} \quad \mathcal{E}_1^\vee \simeq \mathcal{M}_1 \oplus \mathcal{S}_1$$

as given by the statement (1) in Proposition 3.4.6. As $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{O}$ by Proposition 3.4.2, we obtain

$$\mathcal{E} \simeq \mathcal{M}_1^\vee \oplus \mathcal{S}_1^\vee \oplus \mathcal{O}. \quad (3.42)$$

Since \mathcal{S}_1^\vee is a direct summand of \mathcal{E} , we have

$$\mu_{\min}(\mathcal{S}_1^\vee) \leq \mu_{\max}(\mathcal{E}) = 0$$

by the condition (vii) in Proposition 3.4.1. Hence (3.42) yields

$$\mathrm{rk}(\mathcal{E}^{\geq \mu_{\min}(\mathcal{S}_1^\vee)}) \geq \mathrm{rk}(\mathcal{S}_1^\vee \oplus \mathcal{O}) > \mathrm{rk}(\mathcal{S}_1^\vee) \quad (3.43)$$

Moreover, as $\mathcal{E}_1 \not\simeq \mathcal{Q}$, Proposition 3.4.9 and the second equality in (3.39) together imply

$$\mathrm{rk}(\mathcal{S}_1^\vee) = \mathrm{rk}(\mathcal{F}^{>\mu_{\min}(\mathcal{S}_1^\vee)}). \quad (3.44)$$

Since $\mu_{\min}(\mathcal{S}_1^\vee)$ is a slope of \mathcal{E} by (3.42), it is not a slope of \mathcal{F} by the condition (iv) in Proposition 3.4.1. Hence we have

$$\mathrm{rk}(\mathcal{F}^{>\mu_{\min}(\mathcal{S}_1^\vee)}) = \mathrm{rk}(\mathcal{F}^{\geq \mu_{\min}(\mathcal{S}_1^\vee)}). \quad (3.45)$$

Now we combine (3.43), (3.44) and (3.45) to obtain

$$\mathrm{rk}(\mathcal{E}^{\geq \mu_{\min}(\mathcal{S}_1^\vee)}) > \mathrm{rk}(\mathcal{F}^{\geq \mu_{\min}(\mathcal{S}_1^\vee)}).$$

However, this is impossible because of the condition (i) in Proposition 3.4.1 and Proposition 3.1.5. We thus complete the proof by contradiction. \square

Since $c_{\mathcal{Q}, \mathcal{F}}(\mathcal{Q}) = 0$ by Definition 3.3.1, we deduce Proposition 3.4.1 from Proposition 3.4.8, Proposition 3.4.9 and Proposition 3.4.10. This concludes our proof of Theorem 3.1.1.

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